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DECISION MAKING METHOD USED IN THE ABSENCE OF CLEARLY IDENTIFIABLE RULES

The present invention pertains to a decision making method used in the absence of clearly identifiable rules.

The rule-based approach is very widely used in numerous expert systems. It allows experts to enter their knowledge in the form of "rules of the trade" as naturally as possible. The rule-based approach allows the expert to provide his expertise directly in explicit and perfectly clear form.

Decision trees are much used to model the making of a decision from among a finite set of alternatives ("alternative" signifying in the present description one of the possibilities offered by a choice). Their main benefit is that they are entirely comprehensible to an expert. A decision tree may be represented as a set of rules. The difficulty is to take account of the inaccuracies and uncertainties of the knowledge of the expert in these decision trees. The inaccuracies and uncertainties are classically modeled by virtue of the use of fuzzy logic. If one considers the following rule "If $R_1 \geq \alpha_1$ and $R_2 \geq \alpha_2$ then $z \in C$", then this. This amounts to saying that R_1 is greater than or equal to α_1, and likewise for R_2 with α_2. Inaccuracies and uncertainties are not the only phenomena which are worthy of being modeled. According to the above rule, we have $z \in C$ as soon as $R_1 \geq \alpha_1$ and $R_2 \geq \alpha_2$. Now, obviously, it may happen that numerous practical cases exist in which z ought also to belong to C when R_1 is slightly less than α_1 but R_2 is sufficiently greater than α_2. In this case, one actually expects a good value of the variable R_2 to compensate for a poor value of R_1. Obviously the same holds between a good value of the variable R_1 and a poor value of R_2.

An object of the invention is to be able to model compensation phenomena.

The difficulty is to take account in the decision trees of inaccuracies and uncertainties, as well as compensatory phenomena.

Returning to the rule "If $R_1 \geq \alpha_1$ and $R_2 \geq \alpha_2$ then $z \in C$". The consideration of the uncertainties and inaccuracies in the standard rules is conventionally done by virtue of fuzzy logic. This amounts to saying that the condition R_1 may be more or less greater than or equal to α_1 may hold to a greater or lesser extent (with a certain degree) and likewise for R_2 and α_2. One then introduces a fuzzy set V_1 which is nothing other than a function which, with a value of R_1 associates a degree between 0 and 1. This degree equals 0 (that is to say $V_1(R_1)=0$) if the condition $R_1 \geq \alpha_1$ is not satisfied at all, and this degree equals 1 (that is to say $V_1(R_1)=1$) if the condition $R_1 \geq \alpha_1$ is entirely satisfied. In the literature there exist numerous ways of transforming a standard rule into a fuzzy rule. To each way there corresponds an interpretation of the rule. For example, for so-called "certainty" fuzzy rules, this gives: "the more R_1 is greater than α_1 and the more R_2 is greater than α_2, then the more certain it is that $z \in C$".

The above rule may be written in fuzzy form, in the following generic manner: "If $V_1(R_1)$ AND $V_2(R_2)$ are large, then $z \in C$". To model the compensatory phenomena, the AND connector in the rule must be extended. In the literature, the extension of the conjunction and disjunction connectors exists. By replacing the AND connector by a generic connector denoted $\otimes$, we obtain "If $V_1(R_1)$, $\otimes$ $V_2(R_2)$is large then $z \in C$". To model more particularly compensation, one uses a connector of the average type. One then more particularly writes "If $F(V_1(R_1),\ V_2(R_2))$ is large then $z \in C$", where F is a function that will be explained hereinbelow. The

compensatory fuzzy rule that has been described with two variables in the premises generalizes to any number of variables. This gives: "If F $(V_1(R_1), V_2(R_2),\ldots,V_n(R_n))$ is large, then $z\in C$". The number $F(V_1(R_1),\ldots,V_n(R_n))$ corresponds to the degree of compensation between 0 and 1. It describes the degree to which compensation occurs and hence the degree to which the rule must be triggered. The concept of degree (and in particular degree of compensation $F(V_1(R_1),\ldots,V_n(R_n))$ comes back to the concept of bounded unipolar scale modeling a concept whose converse does not exist and whose degree admits a maximum value, such as is the case for example for satisfaction. The degree is therefore typically modeled in a scale [0,1]. On the other hand, the concept of compensation is based on the concept of bipolar scale (modeling a concept and its converse, such as for example attractivity and repulsion) since, in any compensation phenomenon, there are necessarily positive aspects which compensate for the negative aspects. The utility functions $V_i(R_i)$ must therefore correspond to such scales. One sees therefore that the function F has as argument values belonging to a bipolar scale (the utilities $V_i(R_i)$) and returns a value belonging to a bounded unipolar scale (the degree of compensation). Consequently, there must therefore exist inside F a function T making it possible to go from a bipolar scale to a bounded unipolar scale. For any values $u_1,\ldots,u_n$ of its arguments, F may therefore be written $F(u_1,\ldots,u_n)=T(H(u_1,\ldots,u_n))$, where H is an aggregation function such as those used in the multicriteria decision aid realm. The function H models compensation. It is said to be compensatory in the sense that $H(u_1,\ldots,u_n)$ lies between the smallest value among the u_i and the largest value among the u_i. An example of a function H is typically the weighted sum: $H(u_1,\ldots,u_n)= \Sigma_{i\in(1,\ldots,n)}\alpha_1\, u_i$.

As we shall detail later, in any compensatory phenomenon, good aspects compensate for poor ones. The

good aspects are the variables such that $V_i(R_i)$ is large while poor aspects are the variables such that $V_j(R_j)$ is small. Typically, we have $R_i \geq \alpha_1$ for the good variables, and $R_j \leq \alpha_j$ for the bad variables. Certain variables R_j may therefore lie above the thresholds α_j provided that the other variables are sufficiently below the thresholds. We will then say that the limit α_j is not strict.

The concept of fuzzy decision tree exists (cf. J.M. Adamo "Fuzzy decision trees", Fuzzy Sets & Systems, vol. 4, pp. 207-219, 1980). In the literature, the determination of fuzzy decision trees is done in general by learning techniques. These techniques do not make it possible to introduce fuzziness into an already existing decision tree. Moreover, they do not handle compensatory phenomena.

In a standard fuzzy rule of the type "If $U_\alpha(x)$ and $U_\beta(y)$ are large, then $z \in C$", the fuzzy sets U_α and U_β are directly palpable to an expert, so that he will be capable of determining them explicitly. This is no longer directly the case in compensatory rules, since the V_i are perceived solely through the function F. It is therefore very difficult, or even impossible, for an expert to provide the values of the V_i and of F directly. This is why no procedure making it possible to do this actually exists.

There exist a certain number of methods that are relatively classical in fuzzy logic which indirectly make it possible to model compensatory phenomena. The first relates to the use of so-called "conjunctive" fuzzy rules (cf. E.H. Mamdani "Application of fuzzy logic to approximate reasoning using linguistic systems" IEEE Transactions on Computers, No. 26, pp. 1882-1191, 1977). This involves "tiling" the set of possible values of each variable R_i by a series of fuzzy sets $A_{i,1}, \ldots, A_{i,p}$. One then creates a rule for

each combination of the fuzzy sets: "If $R_1 \in A_{1,k1}$ and ... and $R_n \in A_{n,kn}$ then $z \in C$" for all $k_1,...,k_n$. Each combination of the $k_1,...,k_n$ provides a different a priori value of C. The fuzzy rule based approach using an aggregation function F consists in describing the compensation globally by virtue of a mathematical function, while this approach consists in describing the compensation point by point (that is to say for any n-tuple $k_1,...,k_n$). The major drawback of this approach is "combinatorial explosion" since a rule must be made explicit for all the possible combinations of $k_1,...,k_n$. Moreover, the conjunctive rules have an interpretation which does not correspond to an implication between the premise conditions and the conclusion, but just to the observation of something which has occurred (cf. D. Dubois & H. Prade, "What are fuzzy rules and how to use them", Fuzzy Sets & Systems, No. 84, pp. 169-185, 1996). This approach is less relevant than that using the function F.

The second known method is interpolation between rules (cf. D. Dubois & H. Prade, "On fuzzy interpolation", Int. Journal of General Systems, No. 28, pp. 103-114, 1999). One considers for example two rules which are applied to different values of the variables under the premises: " If $R_1 \in A_1$ and ... and $R_n \in A_n$ then $z \in C$" and "If $R_1 \in B_1$ and ... and $R_n \in B_n$ then $z \in C'$". The interpolation between these two rules makes it possible to create rules which will be applied to the intermediate values between the A_i and the B_i. The concatenation of all these rules will have a similar effect to the approach using an aggregation function F. On the other hand, the way in which the global compensatory rule is obtained is sidetracked. The consequences of the interpolation may go beyond what the expert initially desired. According to the invention, it is preferable to aid the expert to reason directly as regards compensation.

The decision making method in accordance with the

invention is a method according to which one establishes decision making rules comprising at least two variables for each of which at least one limit is not strict, and it is characterized in that one formally introduces a compensation condition into the nonclearly identifiable rules, that one determines, for each parameter of a compensatory condition, at least one particular point belonging to a compensation boundary and connected with the parameter, that one deduces therefrom the value of the parameters, that one applies the set of rules and that one deduces the decision therefrom. It will be noted that the fact that a limit is not strict signifies that the conditions on the corresponding thresholds may be violated.

According to a second characteristic of the invention, the compensation is of binary nature, and there is just one single compensation boundary.

According to a third characteristic of the invention, the conditions in the premises are rendered fuzzy by the expert, the compensation may hold to a greater or lesser extent, there are two compensation boundaries, the application of the rules makes it possible to calculate a degree of possibility regarding the set of possible alternatives, and one must interpret the final distributions of possibility so as to deduce the decision therefrom.

According to a fourth characteristic of the invention, the compensation condition is written as the aggregation by a sum, which is advantageously a simple unweighted sum, of utility functions on each variable, the utility functions are piecewise affine, an expert provides the abscissa of the points delimiting the affine parts, and the parameters of the compensation condition are the ordinates of these points.

According to a fifth characteristic of the invention,

the expert provides as relative values with respect to the extreme values the ordinates of the utility functions for all points delimiting the affine parts except for the two extreme points and the threshold, the utility at the threshold is zero and the parameters of the compensation condition are the ordinates of the utility functions for the extreme points.

According to a sixth characteristic of the invention, the utility at the threshold is zero and the parameters of the compensation condition are the ordinates of the utility functions for all points delimiting the affine parts except for the threshold.

According to a seventh characteristic of the invention, the particular points are such that all their coordinates according to the variables except one are equal to one of the values delimiting the affine parts of the utility functions, one requests the expert to provide the value according to the nonfixed coordinate so that the particular point is situated exactly on a compensation boundary, one determines a characteristic point for every variable and every value delimiting the affine parts of the utility function on this variable such that the coordinate of the characteristic point along the variable is equal to the value and such that the ordinate of this value is a parameter (that is to say is unknown), the relations that one has on the characteristic points culminate in a set of system equations whose unknowns are the parameters, and one solves this system with a classical procedure.

According to an eighth characteristic of the invention, the expert determines for each variable the type of compensation to which it belongs, this provides a set of equations and of inequalities to which are appended the equations arising from the characteristic points, and one solves this system according to a classical procedure.

According to a ninth characteristic of the invention, all the variables correspond to a compensation of the type for which, for each variable R_i, there exists a value of R_i above or below which no more compensation is possible regardless of the value according to the other variables, that the expert provides as relative values with respect to the extreme values the ordinates of the utility functions for all points delimiting the affine parts except for the two extreme points and the threshold, that the utility of the threshold is zero, that the parameters of the compensation condition are the ordinates of the utility functions for the extreme points, that fuzziness is introduced, that the conditions in the premises are rendered fuzzy by the expert, that the compensation may hold to a greater or lesser extent, that the characteristic points are such that the component along a well-satisfied variable corresponds to the maximum value along this variable, that the component along a poorly satisfied variable is free, that one asks the expert to provide the value along the free coordinate (not fixed) so that the particular point is situated exactly on a compensation boundary and that all the other components are fixed at the thresholds.

According to a tenth characteristic of the invention, the rule base corresponds to a decision tree.

According to an eleventh characteristic of the invention, the rule base corresponds to a decision tree, and a single alternative is entirely possible in the final distribution of possibilities (this is the hypothesis H described hereinbelow).

According to a twelfth characteristic of the invention, one reveals in the decision tree the pairs of complementary conditions, including the compensation conditions, one processes the complementary conditions

at the same time while separating the kernel of their fuzzy set by a very small number.

According to a thirteenth characteristic of the invention, one commences by formally introducing compensation, then one formally introduces fuzziness, then one specifies the noncompensatory fuzzy conditions, and finally one specifies the compensatory fuzzy conditions.

The present invention will be better understood on reading the detailed description of a mode of implementation, taken by way of nonlimiting example and illustrated by the appended drawing, in which:

figure 1 is an exemplary simplified decision tree serving to explain the invention,

figure 2 is a chart plotted in the plane of the two variables of the tree of figure 1,

figure 3 is a chart like that of figure 2, in which the compensations according to the invention have been introduced,

figures 4 to 6 are charts of fuzzy functions used by the invention,

figures 7 to 13 are charts in the plane of the two variables of a decision tree, in which various compensations, in accordance with the invention, have been introduced, and

figures 14 to 16 are charts of utility functions implemented by the invention.

The general characteristics of the method of the invention will first be described succinctly. Thus, we start from the reading of a decision tree in the form

of standard rules. The following phases describe the process for taking account of inaccuracies and uncertainties, as well as compensatory phenomena.

- P1 - introduction of compensation. The explanation of the decision tree in the form of rules shows that we culminate in a partition of the space of variables into various zones, each zone corresponding to the values of the variables leading to a particular alternative. The introduction of a compensation phenomenon induces a modification of the boundary between two zones. We must therefore determine initially to the boundary between which alternatives the compensation chiefly pertains. According to the invention, we determine the zone in which the compensation induces a domain reduction. We then add a conjunction compensation term to the premise conditions of the rule defining this zone. On the other hand, the neighboring zone will get bigger by the part removed from the previous zone. We must then add a disjunction compensation term to the premise conditions of the rule defining this neighboring zone. This compensation term corresponds to the complement of the initial compensation term.
- P2 - introduction of fuzziness. As stated hereinabove, the premise conditions of the decision tree form a partition of the space of variables. The fact that a partition is obtained implies that whenever a condition is found, its complement is necessarily found somewhere in the same decision tree. Two conditions are complementary if, regardless of the value of the variables, one and only one condition out of these two conditions is entirely true. This corresponds to a hypothesis H described later. In order to introduce fuzziness, the invention proposes that each pair of complementary conditions be handled

conjointly so that the hypothesis H is satisfied. Moreover, under the premise condition of the rules, the conjunction and disjunction operators are transformed into minimum and maximum respectively.

- P3 - specification of fuzziness in the noncompensatory conditions. In order for the hypothesis H to hold, the values for which the fuzzy conditions are entirely satisfied correspond to the values for which the nonfuzzy conditions are satisfied. To determine the values for which the fuzzy conditions are no longer satisfied at all, we can ask the expert questions. As a minimum, it is sufficient to ask a single question per condition.
- P4 - specification of fuzziness under the compensatory conditions. This forms the subject of a separate process described hereinbelow.

It is not necessary to carry out these steps in the order indicated previously. The order described hereinabove is a preferred example. The relations of precedence between the various phases are: P4 comes after P1 and P2; P3 comes after P2. Moreover, it is perfectly possible to introduce fuzziness but not compensation into a decision tree. In this case, phases P2 and P3 are sufficient. Likewise, it is perfectly possible to introduce compensation into the fuzziness in a decision tree. Phases P1 and P4 are then sufficient. It is in fact possible to use the process P4 to specify the compensation found in a nonfuzzy decision tree.

Characteristics 10 to 13 of the invention pertain to the explanation of compensation in a decision tree. Characteristics 11 to 13 of the invention are more particularly concerned with a decision tree in which one wishes to introduce fuzziness while satisfying the hypothesis H. Characteristic 12 of the invention

corresponds to step P2 and consists in particular in handling the pairs of complementary conditions at the same time. Characteristic 13 of the invention consists in using steps P1 to P4 to introduce fuzziness and compensation into a decision tree.

It should be noted that step P4 describes on its own a process which can be considered separately. Specifically, P4 provides a process for explaining a compensatory phenomenon in a fuzzy rule. P4 taken separately can therefore serve to specify the compensation in a condition of the compensation type occurring as a premise of a fuzzy rule.

Characteristics 1 to 9 of the invention pertain to the explanation of compensation in a rule alone, that is to say step P4 taken alone.

The novelty of the method of the invention pertains firstly to the process corresponding to phase P4 (specification of the compensatory fuzzy conditions), and secondly to the fact that introducing fuzziness in a certain manner (by considering the pairs of complementary conditions) so as to ensure that the result (the alternative chosen) satisfies certain properties.

The tricky point in the process consists in formalizing the compensation. In the space of variables, fuzzy compensation is characterized by three zones: one in which we compensate perfectly, another in which we do not compensate at all, and in the middle a zone in which we compensate a little with a degree of compensation lying between 0 and 1. For a point situated in the intermediate zone, the expert will not be capable of determining the degree with which compensation is permitted for this point. We may on the other hand ask the expert which zone a point belongs to. The points for which we deduce the most information

belong to the boundary between two zones. Specifically, it is easy to see that each boundary is characterized as a certain level curve of $H(V_1(R_1), V_2(R_2), \ldots, V_n(R_n))$, that is to say the set of values of variables $R_1, \ldots, R_n$ for which $H(V_1(R_1), V_2(R_2), \ldots, V_n(R_n))$ is equal to a certain value, while the fact of belonging to a zone just gives an inequality in $H(V_1(R_1), \ldots, V_n(R_n))$ - this being less informative. The process therefore consists in asking the expert to specify a certain number of points situated on the boundary between the "we compensate perfectly" zone and the "we compensate a little" zone, and on the boundary between the "we do not compensate at all" zone and the "we compensate a little" zone. The knowledge of these points will make it possible to determine the parameters of the compensation.

In order to determine the utility function V_i defined on the variable R_i, we characterize V_i by a set of discrete points of R_i. This set is denoted $\mathfrak{R}_i$. It is then sufficient to determine the value $V_i(x_i)$ of the utility function at each point x_i of this set $\mathfrak{R}_i$ to fully specify V_i. For example, V_i may be affine between these points.

The method of the invention consists in questioning the expert on a set of singular points of the compensation belonging to the boundary between two zones. These singular points are points all of whose coordinates except one belong to the sets $\mathfrak{R}_i$. We then ask the expert for which value of the last variable is the singular point situated exactly on the boundary between two zones. Such a singular point denoted $R^i(R_i)$ is such that for all $k \neq i$ its component in the variable k belongs to the set $\mathfrak{R}_k$ and its component in the variable i equals R_i. The values of $R^i(R_i)$ along the components other than i are such that when R_i varies, the point $R^i(R_i)$ necessarily cuts one of the two boundaries. If this is the boundary between the "we compensate entirely" zone

and "we compensate a little" zone, we obtain this value of R_i by asking up to what value of R_i or onward of what value of R_i - depending on whether the utility function V_i is increasing or decreasing - we compensate entirely for the point $R^i(R_i)$. If this is the boundary between the "we do not compensate at all" zone and the "we compensate a little" zone, we obtain this value of R_i by asking onward on what value of R_i or up to what value of R_i - depending on whether the utility function V_i is increasing or decreasing - we no longer compensate at all for the point $R^i(R_i)$. For each variable j and each point x_j of $\mathfrak{R}_j$, we search for a point $R^i(R_i)$, satisfying the previous hypotheses, such that its component along the variable j equals x_j. For the value R_i for which $R^i(R_i)$ is located at a boundary, we obtain an equation (corresponding to the fact that $H(V_1(R^i(R_i)_1),\ldots,V_n(R^i(R_i)_n))$ is equal to a certain value) in which $V_j(x_j)$ is found. By doing this for each variable j and each point x_j of $\mathfrak{R}_j$, we thus obtain a system of equations that we can solve in a classical manner.

These equations can also be supplemented with a few inequalities pertaining in particular to the type of compensation desired. We have in fact identified three typical compensation behaviors (denoted R1, R2 and R3 hereinafter). These behaviors make it possible to give a clear definition to the key values of each variable R_i. The relations that are obtained may be solved either directly or via a standard linear program or a linear integer program.

As was indicated above, for each variable j and each point x_j of $\mathfrak{R}_j$, we search for a point $R^i(R_i)$, satisfying the previous hypotheses, such that its component along the variable j equals x_j. We therefore search for the values of $R^i(R_i)$ in the components other than i and j such that $R^i(R_i)$ cuts at least one of the two level curves when R_i varies. In the favorable cases, we can

fix the values of the components of $R^i(R_i)$, i and j excluded, at values whose utilities are already known, while ensuring that $R^i(R_i)$ necessarily cuts one of the two boundaries for a value of R_i. In the other cases we must use values of the components of $R^i(R_i)$, i and j excluded, whose utilities we do not yet know. We cannot then be sure that $R^i(R_i)$ necessarily cuts one of the two level curves since, as a matter of fact, this property depends on the utilities along the components other than i and j which are not yet known. In this case, we can choose the other components which maximize a sort of probability of crossing a level curve.

To finish, we will briefly explain how compensation (without fuzziness) is introduced into a nonfuzzy rule. Here we make reference to the second characteristic of the invention as well as to those which stem therefrom. The fact of culminating at the end of the process in an unequivocal binary decision ("we compensate" or "we do not compensate") implies that there are only two zones, and hence a single boundary instead of two as previously. Moreover, the function T returns 1 (in this case, "we compensate") if its argument is greater than a certain value, and 0 (in this case, "we do not compensate") otherwise. We then proceed as previously, except that we have only a single level curve. Therefore, since we have a single absolute reference level (the value of the level curve), the results will be given to within a homothety (with respect to the value). This will be sufficient to be able to make the comparison with respect to this value with no possible ambiguity.

The present invention will be described hereinbelow with reference to an exemplary decision tree. It is simple, but representative of the phenomena that may arise. The example consists of an end-of-year examination. The examination is composed of two tests. The results according to the two tests are denoted R_1

and R_2. As a function of the results obtained, we have three possible states: *Accepted* (the end-of-year examination was passed and the student is accepted), *Resit* (the results are not sufficient to accept the student straightaway, but sufficiently correct to permit him to take an oral resit), and *Refused* (the examination was failed, and the student is refused). The decision tree represented in figure 1 indicates the situation of the student as a function of his results. This decision tree stipulates that R_1 and R_2 must be greater than or equal to 10. If these two conditions are both satisfied, the student is accepted. If not, we look to see whether these two results are both greater than or equal to 8. If they are, the student is passed, if not, he is refused.

In the plane of the variables R_1 and R_2, the zones demarcating the *Accepted, Resit* and *Refused* situations are represented in figure 2, which faithfully conveys in this plane the decision tree of figure 1.

A process making it possible to take account of inaccuracies, uncertainties and compensatory phenomena in decision trees will now be described.

Decision trees are much used to model the making of a decision from among a finite set of alternatives $C=\{C_1,\ldots,C_h\}$. Their main benefit is that they are entirely comprehensible to an expert. The introduction of fuzziness into a tree leads, during the traversal of this decision tree, to the fact that we no longer culminate in a single alternative C_m but in a distribution of possibilities $\wp_C$ over the set of possible alternatives. Thus, for each alternative $C_m \in C$, we determine the degree of possibility $\wp_C(C_m)$ (lying between 0 and 1) that C_m is the decision to be made. The interpretation of this distribution of possibilities is then done based on the theory of possibilities.

To introduce fuzziness and compensation into each decision tree, the invention makes provision to start from the standard decision tree. We shall firstly explain how to introduce compensation, and more precisely the form that compensation must take (if there is reason to introduce compensation). We will then describe how to introduce fuzziness.

We begin by introducing compensation into a tree that has not yet been rendered fuzzy. We must start from the writing of a decision tree in the form of standard rules giving the conditions of allocation of each alternative. For the example of the examination, this gives:

"*Accepted*" if $R_1 \geq 10$ AND $R_2 \geq 10$

"Resit" if $(8 \leq R_1 < 10$ AND $R_2 \geq 8)$ OR $(R_1 \geq 8$ AND $8 \leq R_2 < 10)$

"Refused" if $R_1 < 8$ OR $R_2 < 8$

As it is difficult to take account of compensation without a suitable tool, no compensation phenomenon is generally made explicit in trees. It may be that there is no reason to introduce it. On the other hand, the explanations hereinbelow relate to the converse case. The explanation of the decision tree in the form of rules does indeed show that we culminate in a partition of the space of variables into various zones, each zone corresponding to the values of the variables leading to a particular alternative. The introduction of compensation induces a modification of the boundary between two zones. We must therefore determine firstly to which boundary between alternatives the compensation chiefly pertains, and secondly to which variables the compensation applies.

Here we return to the example of the examination. The conditions of acceptance in an examination are in general strict and it can be assumed that it is perhaps not desirable to introduce compensation on the

"*Accepted*" alternative. On the other hand, we may ask ourselves whether a student has to be refused if a result is less than 8 even when the other result is excellent. Likewise, we may ask ourselves whether a student who obtained the score 8 according to the two results is nevertheless worthy of going to a resit session. Stated otherwise, we assume that it is relevant to introduce a compensation at the boundary between the "*Resit*" and "*Refused*" alternatives, and that this relates to the variables R_1 and R_2. We will necessarily have to modify the set of values of R_1 and R_2 leading to the "*Resit*" and "*Refused*" alternatives. The boundary between these two alternatives may be modified in two ways:

- The first way consists in restricting the set of values of R_1 and R_2 leading to the "*Resit*" alternative. In the absence of compensation, the worst results possibly being able may lead again to a resit $R_1=8$ and $R_2=8$. Consequently, in this first way of doing things, a student who obtained the score 8 in the two results does not deserve to go to a resit session. We are more demanding, more intolerant. We now assume that the result R_1 is more significant than the result R_2. In this case, we tolerate a relatively poor value of R_1 only if R_2 has a good enough value. Stated otherwise, we allocate the "*Resit*" alternative if a rather good value of the result R_2 (that is to say $R_2<8$) compensates for a relatively poor value of the result R_1 (that is to say for example $8 \le R_1 < 10$). Of course, there are other approaches for restricting the set of values of R_1 and R_2 leading to the "*Resit*" alternative.
- The second way consists in extending the set of values of R_1 and R_2 leading to the "*Resit*" alternative. Let us assume that the result R_1 is more significant than the result R_2. We then decide to pass a poor value of R_2 (that is to say $R_2<8$) when R_1 is good enough. We are less

demanding, and more tolerant. Stated otherwise, we allocate the "*Resit*" alternative if a rather good value of the result R_1 (that is to say $R_1 \geq 8$) compensates for a poor value of the result R_2 (that is to say $R_2 < 8$). This is just one approach to extending the set of values of R_1 and R_2 leading to the "*Resit*" alternative.

Figure 3 shows these two ways of doing things.

These two ways of introducing compensation are very different. Choosing the type amounts to ascertaining whether all the values of R_1 and R_2 leading to the "*Resit*" alternative in the initial decision tree actually deserve to have this alternative allocated. To determine the right way of doing things, we can ask the following question:

If $R_1=8$ and $R_2=8$, do you actually think that the "*Resit*" alternative deserves to be allocated?

This amounts to asking whether the worst values of the variables R_1 leading to the alternative in the initial decision tree actually deserve to culminate in this alternative. If the response is positive, then the second way of doing things is the right one.

It is also possible to mix the previous two ways of doing things by restricting on a certain side the values of R_1 and R_2 leading to the "*Resit*" alternative, and by extending on another side R_1 and R_2 leading to the "*Resit*" alternative, as represented in the chart of figure 3.

We will first examine the case of the introduction of compensation by restriction (intolerance).

Here, compensation restricts the possible values leading to the "*Resit*" alternative. This implies that compensation is a condition which supplements the

already existing conditions. Let us take the example of the examination. The condition of allocating the "*Resit*" alternative without compensation is:

"*Resit*" if ($8 \le R_1 < 10$ AND $R_2 \ge 8$) OR ($R_1 \ge 8$ AND $8 \le R_2 < 10$)

The compensation described previously amounts to restricting the condition $8 \le R_1 < 10$ and $R_2 \ge 8$. In order for the two conditions not to exhibit any mutual intersection, we rewrite this rule:

"*Resit*" if ($8 \le R_1 < 10$ AND $R_2 \ge 8$) OR ($R_1 \ge 10$ AND $8 \le R_2 < 10$)

Since we restrict the values leading to the "*Resit*" alternative, this implies quite simply that we append a further condition to the condition "$8 \le R_1 < 10$' and $R_2 \ge 8$" and that this condition pertains precisely to compensation:

"*Resit*" if ($8 \le R_1 < 10$ AND $R_2 \ge 8$ AND R_2 compensates for R_1) OR ($R_1 \ge 10$ AND $8 \le R_2 < 10$)

Without compensation, the condition of allocation of the "*Refused*" alternative is:

"*Refused*" if $R_1 < 8$ OR $R_2 < 8$

With the introduction of compensation by restriction on the "*Resit*" alternative, we extend the conditions of allocation of the "*Refused*" alternative. Stated otherwise, in addition to the previous conditions, the "*Refused*" alternative is also allocated for values which are no longer allocated to the "*Resit*" alternative when compensation has been introduced. These are the values satisfying the conditions of allocation of the "*Resit*" alternative without compensation, for which compensation must not occur:

$8 \le R_1 < 10$ AND $R_2 \ge 8$ AND R_2 does not compensate for R_1

Hence, in the case of compensation, we have

"*Refused*" if $R_1 < 8$ OR $R_2 < 8$ OR ($8 \le R_1 < 10$ and $R_2 \ge 8$ and R_2 does not compensate for R_1)

The conditions "R_2 compensates for R_1" and "R_2 does not compensate for R_1" are complementary.

We now examine the cases of the introduction of compensation by extension (tolerance).

Here, compensation extends the possible values leading to the "*Resit*" alternative. It is easier firstly to explain the alternative for which the domain is restricted. This is the "*Refused*" alternative. Without compensation, the condition of allocation of the "*Refused*" alternative is:

"*Refused*" if $R_1<8$ OR $R_2<8$

We restrict the condition "$R_2<8$". We therefore append the compensation condition to the condition "$R_2<8$", as previously:

"*Refused*" if $R_1<8$ OR ($R_2<8$ AND R_1 does not compensate for R_2)

Here, the compensation condition is "R_1 does not compensate for R_2". The term "compensation" has a positive connotation. The fact of being able to compensate must therefore culminate in a positive conclusion. Out of the two alternatives "*Resit*" and "*Refused*", it is the allocation of the "*Resit*" alternative which is the most positive. Thus, one speaks of compensation when arriving at the "*Resit*" alternative and of noncompensation when culminating in the "*Refused*" alternative. This is why the compensation condition makes reference to "do not compensate for".

The conditions of allocation of the other alternative are obtained as previously. We therefore have the "*Resit*" alternative if the conditions of allocation of the "*Resit*" alternative without compensation are satisfied or else if we have a lack of noncompensation in the conditions of allocation of the "*Refused*" alternative:

"*Refused*" if ($8 \leq R_1 < 10$ AND $R_2 \geq 8$) OR ($R_1 \geq 8$ AND $8 \leq R_2 < 10$) OR ($R_2 < 8$ AND R_1 compensates for R_2)

We now examine the case of the introduction of fuzziness into a decision tree.

We must start from the reading of a decision tree in the form of standard rules, with, preferably, already, introduction of compensation.

Reasoning with regard to decision trees affords a beneficial property. Specifically, in a decision tree, we know the alternative allocated for all the possible values according to the variables. Moreover, conditions of allocation of each alternative are explicitly available. It therefore transpires that the domains in which each alternative is allocated form a partition of the space of variables. This implies that whenever we find a condition, we necessarily find its complement somewhere in the same decision tree. For example, $R_1 \geq 10$ with $R_1 < 10$.

Firstly, we examine the case of standard conditions. The introduction of fuzziness will consist in rendering the conditions fuzzy. Instead of considering the condition $R_1 < 10$ to be true or false, we define a degree of validity on the condition. For a condition R, we will subsequently denote by U(R) the degree of validity on the condition R. U(R)=1 if the condition R is entirely satisfied, and U(R)=0 if the condition R is not satisfied at all. For example, $U(R_1 \geq 10)$ is the degree of validity of the condition $R_1 < 10$ according to the value of R_1.

Therefore, for each condition we must define a degree of validity for the condition and also for its complement. In standard logic, two conditions R and T are complementary if we have either R or T which is true. In the theory of possibilities, this condition

becomes:

> **<u>Hypothesis H:</u> max(U(R) , U(T)) = 1. Moreover, we have either U(R)=1, or U(T)=1.**

This condition implies that out of the two conditions R and T, one is always entirely satisfied. Moreover, they cannot be entirely satisfied at the same time. Several ways of satisfying the hypothesis H are possible. We describe one of them for the example of the complementary conditions $R_1 \geq 10$ and $R_1 < 10$:

- The condition $R_1 \geq 10$ is entirely satisfied when $R_1 \geq 10$. It is not no longer satisfied at all when R_1 is *much less* than 10. $U(R_1 \geq 10)$ therefore has the shape given in figure 4.
- The condition $R_1 < 10$ is entirely satisfied if $R_1 < 10$. Thus, we must have $U(R_1 < 10) < 1$ for $R_1 = 10$. Hence, for $R_1 = 10$, only the condition $R_1 \geq 10$ will be entirely satisfied. In order for these conditions to hold, we set $U(R_1 < 10)$ equally to 1 if and only if $R_1 \leq 10 - \varepsilon$, where ε is a very small value. For the value $10 - \varepsilon < R_1 < 10$, neither of the two conditions $R_1 \geq 10$ and $R_1 < 10$ holds entirely. These are the only values of R_1 for which the property H does not hold.

In practice, ε will have to be less than the numerical accuracy in the variable R_1. By proceeding in this way, as the value $R_1 = 10$ is attainable, we are sure that the values $10 - \varepsilon < R_1 < 10$ are not numerically attainable. Consequently, the property H holds for all the attainable values of R_1.

Once the various utilities have been defined, we simply obtain the degree of possibility regarding the various conditions appearing in the decision tree. When there are several premise conditions composed with the AND and OR operators, we do the same, replacing AND by a minimum (denoted $\wedge$) and OR by a maximum (denoted $\vee$). Thus, for the example of the examination in the absence

of compensation this gives:

$$\wp(\textit{Accepted}) = U(R_1 \geq 10) \wedge U(R_2 \geq 10)$$

$$\wp(\textit{Resit}) = (U(R_1 \geq 8) \wedge U(R_1 < 10) \wedge U(R_2 \geq 8)) \vee (U(R_1 \geq 8) \wedge U(R_2 \geq 8) \wedge U(R_2 < 10))$$

$$\wp(\textit{Refused}) = U(R_1 < 8) \vee U(R_2 < 8)$$

where $\wp$(Accepted), $\wp$(Resit) and $\wp$(Refused) are the degrees of possibility associated with the three possible alternatives.

We have therefore just shown that the introduction of fuzziness into decision trees amounts to rendering each condition fuzzy, the remainder of the work being trivial.

Reasoning conjointly with regard to a condition and its complement ensures that the condition H holds.

We now examine the case of compensation conditions. The compensation conditions are of the type "R_2 compensates for R_1" and "R_2 does not compensate for R_1". We note that these conditions are complementary. We will therefore proceed in exactly the same way as previously. We will firstly examine the case of an expression of a fuzzy condition of the compensation type. We wish to specify the expression for the utility $U(R_2$ compensates for $R_1)$ of a compensation condition of the type "R_2 compensates for R_1". We adopt the very general framework in which the variables to be taken into account in the compensation are $R_1, \ldots, R_n$. We write $N=\{1,\ldots,n\}$ and $R=(R_1,\ldots,R_n)$. The vector R stripped of its R^{ith} component, that is to say $(R_1,\ldots,R_{i-1}, R_{i+1},\ldots, R_n)$, is denoted R_{-i}. To determine whether the compensation should or should not be allocated, we calculate a utility $U(R)$ which aggregates partial utilities $V_i(R_i)$ according to the different variables. $U(R)$ may then be written in the form of $U(R)=H(V_1(R_1),\ldots, V_1(R_1))$, where H is an aggregation function. Since we wish to model compensation, H is typically a weighted sum $H(u) = \Sigma_{i\in\{1,\ldots,n\}} \alpha_i u_i$. This

gives $U(R)=\Sigma_{i\in\{1,\ldots,n\}}$ α_i $V_i(R_i)$. In the case of compensation, certain variables compensate for other variables. This implies that the variables which compensate have satisfactory values while the compensated variables have unsatisfactory values. We therefore see that the utilities V_i must convey satisfactory values and also unsatisfactory values. Moreover, the neutral element (neither good nor bad) exists. It corresponds to a threshold denoted S_i on the variable i. We therefore see that the utilities V_i have the meaning of a bipolar ratio scale in which the positive values correspond to the satisfactory values, the negative values correspond to the unsatisfactory values, and the value 0 corresponds to the neutral element (the threshold). We have $V_i(s_i)=0$. When the values according to all the variables are satisfactory (that is to say above the thresholds), then we of course compensate entirely, since there is no unsatisfactory value apt to prevent or attenuate the compensation. In this case, since all the $V_i(R_i)$ are positive, the global utility U(R) is strictly positive. When all the variables are equal to the thresholds, then we also compensate entirely. We have $U(s_1,\ldots,s_n)=0$. On the other hand, if all the variables except one are equal to the thresholds and if this variable is unsatisfactory, then we no longer compensate entirely. In this case, the utility of this variable is negative and the other utilities are zero. Since H is a compensatory function, its value lies between the smallest value among its arguments and the largest. From this we deduce that U(R) is negative in this case. We therefore see that the value $U(R)=0$ is the smallest value from which we compensate entirely. Stated otherwise, we compensate entirely if and only if $U(R)\geq 0$. Likewise, below a certain negative value, we will no longer compensate at all. On the other hand, there is no reason to take a particular value here. We arbitrarily choose the value -1. Hence we do not

compensate at all if and only if $U(R) \leq -1$. When $-1 < U(R) < 0$, we compensate a little.

In the expression for U(R), we note that V_i is always present in the product $\alpha_i V_i$. Now, in contradistinction to the fuzzy sets regarding the conditions in the noncompensatory fuzzy rules, V_i does not comprise any absolute reference level, except for the threshold s_i enabling V_i to be normalized. Besides, we have arbitrarily chosen the value -1 as limit of "we do not compensate at all" since in actual fact it is not possible to normalize V_i. It is therefore not relevant to separate the weight α_i of V_i. For this reason, in what follows, the product $\alpha_i V_i$ will be simply denoted V_i. This implies that the weight α_i is included within V_i. Consequently, in what follows, U(R) will be equal to $\Sigma_{i \in \{1,\ldots,n\}} V_i(R_i)$. As previously, we compensate entirely if $U(R) \geq 0$, and we no longer compensate at all if $U(R) \leq -1$.

We must transform the value provided by U(R) so as to determine a degree of compensation lying between 0 and 1. We then apply a function T to U(R). Thus, the degree of compensation will equal T(U(R)). In accordance with the foregoing, we compensate entirely if $U(R) \geq 0$. Hence, as the degree of compensation equals 1 when we compensate entirely, $T(u)=1$ if $u \geq 0$. Moreover, we do not compensate at all if $U(R) \leq -1$. Hence, as the degree of compensation equals 0 when we do not compensate at all, $T(u)=0$ if $u \leq -1$. The function T is given in figure 5. Hence the utility of the compensation condition is U(compensation in $R_1, \ldots, R_n$) = T(U(R)).

The condition complementary to "compensation in $R_1, \ldots, R_n$" is "noncompensation in $R_1, \ldots, R_n$". The utility of the complementary condition is simply U (noncompensation in $R_1, \ldots, R_n$) = T'(U(R)), where the two functions T and T' must be mutually complementary in the sense of the condition H. There is therefore an ε

which separates the kernel of T and T' (that is to say the values of u such that T(u)=1 and the values of u such that T'(u)=1), as shown in figure 5.

We will now examine the case of the introduction of compensation by restriction (intolerance). We return to the example of the examination. The conditions of allocation of the "*Resit*" and "*Refused*" alternatives are:

"*Resit*" if ($8 \le R_1 < 10$ AND $R_2 \ge 8$ AND R_2 compensates for R_1) OR ($R_1 \ge 10$ AND $8 \le R_2 < 10$)

"*Refused*" if $R_1 < 8$ OR $R_2 < 8$ OR ($8 \le R_1 < 10$ and $R_2 \ge 8$ and R_2 does not compensate for R_1)

The conditions "R_2 compensates for R_1" and "R_2 does not compensate for R_1" are complementary. To introduce fuzziness into these conditions, we proceed as described previously, that is to say by replacing AND by $\wedge$ and OR by $\vee$:

$$\wp(\textit{Resit}) = (U(R_1 \ge 8) \wedge U(R_1 < 10) \wedge U(R_2 \ge 8) \wedge U(R_2 \text{ compensates for } R_1)) \vee (\ U(R_1 \ge 10) \wedge U(R_2 \ge 8)\ U(R_2 < 10)\ )$$

In the term "$U(R_1 \ge 8) \wedge U(R_1 < 10) \wedge U(R_2 \ge 8) \wedge U(R_2$ compensates for $R_1)$", the first three conditions make it possible to fix the limits of the compensation, the latter being specified beyond these conditions. The first three conditions ensure the fact that the degree of possibility $\wp$(*Resit*) is gradual. The last condition is concentrated on the compensation and is not concerned with the smooth transitions outside of these three conditions. We proceed in the same manner for the "*Refused*" alternative:

$$\wp(\textit{Refused}) = U(R_1 < 8) \vee U(R_2 < 8) \vee (\ U(R_1 \ge 8) \wedge U(R_1 < 10) \wedge U(R_2 \ge 8) \wedge U(R_2 \text{ does not compensate for } R_1)\ )$$

In accordance with what we showed previously, we have:

$$U(R_2 \text{ compensates for } R_1) = T(V_1(R_1) + V_2(R_2))$$

and

$$\mathbf{U(R_2 \text{ does not compensate for } R_1) = T'(\ V_1(R_1) + V_2(R_2))}$$

In the case of the introduction of compensation by extension (tolerance), we proceed in exactly the same manner. We will see what this gives in the example of the examination. The conditions of allocation of the "*Resit*" and "*Refused*" alternative are:

"*Resit*" if $(8 \le R_1 < 10 \text{ AND } R_2 \ge 8)$ **OR** $(R_1 \ge 8 \text{ AND } 8 \le R_2 < 10)$ **OR** $(R_2 < 8 \text{ AND } R_1$ **compensates for** $R_2)$

"*Refused*" if $R_1 < 8$ **OR** $(R_2 < 8$ **AND** R_1 **does not compensate for** $R_2\ )$

The introduction of fuzziness then gives:

$\wp(\textit{Resit}) = (U(R_1 \ge 8) \wedge U(R_1 < 10) \wedge U(R_2 \ge 8) \wedge U(R_2 \text{ compensates for } R_1)) \vee (U(R_1 \ge 8) \wedge U(R_2 \ge 8)\ U(R_2 < 10)\ ) \vee (U(R_2 < 8) \wedge U(R_1 \text{ compensates for } R_2))$

$\wp(\textit{Refused}) = U(R_1 < 8) \vee (\ U(R_2 < 8) \wedge U(R_1 \text{ does not compensate for } R_2))$

where:

$$\mathbf{U(R_1 \text{ compensates for } R_2) = T(V_1(R_1) + V_2(R_2))}$$

and:

$$\mathbf{U(R_1 \text{ does not compensate for } R_2) = T'(V_1(R_1) + V_2(R_2))}$$

We will now explain the manner in which the fuzzy sets must be specified in a noncompensatory condition.

For a condition R, in order to satisfy the hypothesis H, we have U(R)=1 when the condition R holds in the standard sense (nonfuzzy). We have U(R)=0 when the condition R is no longer satisfied at all. We ask the expert to give the limit point(s) for which R is not satisfied at all. This (these) limit point(s) may also be asked for through the consequences of the fact that U(R)=0 in the decision tree.

We will take the case of $U(R_1 \ge 10)$. Quite often, $U(R_1 \ge 10)$ will be defined by three straight line spans: $U(R_1 \ge 10)=1$ if $R_1 \ge 10$, $U(R_1 \ge 10)=0$ if $R_1 < R_{1,\blacklozenge}$, and $U(R_1 \ge 10)$ is affine

in the middle. In this case, it therefore suffices to provide a unique value $R_{1,\blacklozenge}$. This is the value of R_1 below which $U(R_1 \geq 10)$ is zero the whole time. To obtain the value of $R_{1,\blacklozenge}$, we may ask the expert:

Below what value of R_1^N, do you think that the condition $R_1 \geq 10$ is not satisfied at all?

To make this yet more concrete for the expert, we can refer to a consequence regarding the conclusion of the fact that $U(R_1 \geq 10)=0$. As $\wp$(*Accepted*) = $U(R_1 \geq 10) \wedge U(R_2 \geq 10)$, the fact that $U(R_1 \geq 10)=0$ implies that the "*Accepted*" alternative is impossible. We then may ask the expert the following question:

Below what value of R_1 do you think that the "*Accepted*" alternative should no longer be allocated at all?

When the link between the condition that we are seeking to specify and the alternative is more complex, then the value of the other conditions must be imposed so that the fact that the condition does not hold at all implies that the alternative is no longer possible at all. We then pose a question to the expert making reference to the hypotheses regarding the other conditions.

We therefore see that the expert may reply to only a single question per condition in order to introduce fuzziness (first curve of figure 6). On the other hand, there is nothing to compel us to take $U(R_1 \geq 10)$ to be simply affine between $R_1= R_{1,\blacklozenge}$ and $R_1=10$. This is necessary when the behavior between $R_{1,\blacklozenge}$ and 10 is nonlinear. $U(R_1 \geq 10)$ being interpreted as the possibility regarding the "*Accepted*" alternative when $U(R_2 \geq 10)=1$, it may for example happen that the "*Accepted*" alternative is still almost impossible when R_1 is a little greater than $R_{1,\blacklozenge}$, while the "*Accepted*" alternative is no longer entirely possible at all when R_1 is slightly less than 10, as is the case with the second curve of figure 6.

In such cases, the invention makes provision to construct $U(R_1 \geq 10)$ as a piecewise affine function defined by a few points between $R_{1,\blacklozenge}$ and 10. The value of $U(R_1 \geq 10)$ for these points is determined using a procedure arising out of measure theory and making it possible to construct a difference scale. Such a scale is given to within a translation and a homothety. These two degrees of freedom are fixed by the two conditions $U(R_1 \geq 10)=0$ when $R_1= R_{1,\blacklozenge}$ and $U(R_1 \geq 10)=1$ when $R_1=10$.

Regarding the condition $U(R_1<10)$, the value onwards of which the condition $R_1<10$ no longer holds at all is denoted $R_1^{\blacklozenge}$.

We will now examine the typical compensation behaviors. Let $N=\{1,\ldots,n\}$ be the set of variables to be taken into account in the compensation. The weighted sum characterizing the compensation may be written $U(R)=\Sigma_{i\in\{1,\ldots,n\}} V_i(R_i)$, with the notation $R=(R_1,\ldots,R_n)$. The vector R stripped of its i^{th} component, that is to say $(R_1,\ldots, R_{i-1}, R_{i+1},\ldots,R_n)$, is denoted R_{-i}. We compensate entirely if $U(R)\geq 0$, and we do not compensate at all if $U(R)\leq -1$. We will specify the utility functions V_i a little more. For practical reasons, we assume that the utility functions V_i are bounded. Thus, V_i admits a maximum value V_i^* attained at a point R_i^* : $V_i(R_i^*)=V_i^*$. Likewise, V_i admits a minimum value $V_{i,**}$ attained at a point $V_{i,**}$: $V_i(R_{i,**})=V_{i,**}$. The behavior of V_i between $R_{i,**}$ and R_i^* is not specified for the moment, except for the fact that $V_i(s_i)=0$.

The vector $(s_1,\ldots,s_n)$ belongs to the 0 level curve with U since $U(s_1,\ldots,s_n)=0$. We seek to determine the limits on each variable of the 0 and -1 level curves. We begin with the 0 level curve. Let $i\in N$. We want to know whether, below a certain bad value of this variable R_i, it will no longer ever be possible to compensate for it entirely. We seek the value $R_{i,\#}$ of R_i (if it exists) for which for all R_i such that $V_i(R_i)<V_i(R_{i,\#})$ and for

all R_{-i}, we have $U(R)<0$, and for which for all R_i such that $V_i(R_i) \geq V_i(R_{i,\#})$, there exists R_{-i} such that $U(R) \geq 0$. We therefore seek the value of R_i with $V_i(R_i)$ as negative as possible such that we still compensate entirely. We therefore seek:

$$\text{Min } \{V_i(R_i),\ \exists\ R_{-i} \text{ such that } U(R) \geq 0\}$$

We have $U(R) \leq V_i(R_i) + \sum_{j \in N \backslash i} V_j^*$. Hence if $U(R) \geq 0$, then necessarily $U(R_i, R_{-i}^*) \geq 0$, where $R_{-i}^* = (R_1^*, \ldots, R_{i-1}^*, R_{i+1}^*, \ldots, R_n^*)$. Hence, the above minimum equals:

$$\text{Min } \{V_i(R_i)\ ,\ V_i(R_i) + \sum_{j \in N \backslash i} V_j^* \geq 0\}$$

If this minimum exists, then it is necessarily attained at $R_i = R_{i,\#}$ satisfying:

$$V_i(R_{i,\#}) + \sum_{j \in N \backslash i} V_j^* = 0$$

It is easy to see that $R_{i,\#}$ may very well be equal to $R_{i,**}$. Hence, $V_i(R_{i,\#})$ may attain the value $V_{i,**}$. Hence $V_i(R_{i,\#}) \geq V_{i,**}$. By replacing this condition in the relation satisfied by $R_{i,\#}$, we deduce therefrom that the minimum exists and is attained if and only if:

$$V_{i,**} + \sum_{j \in N \backslash i} V_j^* \leq 0$$

If this condition does not hold, then we compensate entirely, even for infinite values of R_i with $V_i(R_i) \leq 0$.

Case where $V_{i,**} + \sum_{j \in N \backslash i} V_j^* \leq 0$ (with V_i decreasing): this case is illustrated in figure 7.

Case where $V_{i,**} + \sum_{j \in N \backslash i} V_j^* > 0$ (with V_i decreasing): this case is illustrated in figure 8.

We will now examine the characteristics of the -1 level curve. Let $i \in N$. We wish to know whether, onward of a certain bad value of this variable R_i, it will no longer ever be possible to compensate for it at least a little. We seek the value $R_{i,\$}$ of R_i (if it exists) for which for all R_i such that $V_i(R_i) \leq V_i(R_{i,\#})$ and for all R_{-i}, we have $U(R) \leq -1$, and for which for all R_i such that $V_i(R_i) > V_i(R_{i,\#})$ there exists R_{-i} such that $U(R) > -1$. We therefore seek the value of R_i with $V_i(R_i)$ as

negative as possible for which we no longer compensate at all. We therefore seek:

$$\text{Max } \{ V_i(R_i) \text{ , such that } U(R) \leq -1, \forall R_{-i} \}$$

We have $U(R) \leq V_i(R_i) + \sum_{j\in N\backslash i} V_j^*$. Hence for $U(R) \leq -1$ for all R_{-i}, it is necessary and sufficient that $U(R_i,R_{-i}^*) \leq -1$. Hence the above maximum equals:

$$\text{Max } \{ V_i(R_i) \text{ , } V_i(R_i) + \sum_{j\in N\backslash i} V_j^* \leq -1 \}$$

If this maximum exists, then it is necessarily attained at $R_i = R_{i,\$}$ satisfying:

$$V_i(R_{i,\#}) + \sum_{j\in N\backslash i} V_j^* = -1$$

It is easy to see that $R_{i,\$}$ may very well be equal to $R_{i,**}$. Hence, we have: $V_i(R_{i,\$}) \geq V_{i,**}$. By replacing this condition in the relation satisfied by $R_{i,\$}$, we deduce therefrom that the minimum exists and is attained if and only if:

$$V_{i,**} + \sum_{j\in N\backslash i} V_j^* \leq -1$$

Case where $V_{i,**} + \sum_{j\in N\backslash i} V_j^* \leq -1$ (with V_i decreasing): this case is illustrated in figure 9.

Case where $V_{i,**} + \sum_{j\in N\backslash i} V_j^* > -1$: this case is illustrated in figure 10.

In the premises of the rules, the compensation terms are combined via min and max operators with other conditions. Thus, an unbounded behavior (that is to say with a level curve going to infinity) in the compensation may be bounded by other conditions.

Confining ourselves to a single variable R_i, there are three possible behaviors by combining the previous cases:

Case where $V_{i,**} + \sum_{j\in N\backslash i} V_j^* \leq -1$. For V_i decreasing this gives the chart of figure 11.

Case where $-1 < V_{i,**} + \sum_{j\in N\backslash i} V_j^* \leq 0$. For V_i decreasing this gives the chart of figure 12.

Case where $0 < V_{i,**} + \sum_{j\in N\backslash i} V_j^*$. For V_i decreasing this gives the chart of figure 13.

We will now set forth the methodology of explanation of

compensation according to the invention. We adopt the very general framework in which we have n variables denoted $R_i, \ldots, R_n$. It will be recalled here that $U(R_1,\ldots,R_n) = \Sigma_{i\in\{1,\ldots,n\}} V_i(R_i)$. We compensate entirely if $U(R_1,\ldots,R_n)\geq 0$, and we no longer compensate at all if $U(R_1,\ldots,R_n)\leq -1$.

For each $i\in\{1,\ldots,n\}$, the utility function V_i is assumed to be monotonic, that is to say either increasing, or decreasing. The sign of the derivative of V_i is denoted ε_i. We have $\varepsilon_i=1$ if V_i is increasing, and $\varepsilon_i=-1$ if V_i is decreasing. We assume that the compensation pertains to the alternative $C_m\in C$.

In the introduction of compensation, several compensation situations may be possible. In the example cited above (in regard to the introduction of compensation into a decision tree), the compensation introduced is of the type:

R_1 compensates for R_2

This corresponds to the case where the variable R_1 is more important than R_2. In the case where no variable is dominant, both directions of compensation could have been possible:

(R_1 compensates for R_2) OR (R_2 compensates for R_1)

We therefore see that several compensation situations may be permitted jointly. In the general case, we assume that the expert permits a certain number (denoted t) of possible compensation situations:

(the variables of A^+_1 compensate for the variables of A^-_1) OR ...
OR (the variables of A^+_t compensate for the variables of A^-_t)

For all $p\in\{1,\ldots,t\}$, we have $A^+_p \cap A^-_p=\varnothing$ and $A^+_p \cup A^-_p=\{1,\ldots,n\}$. Stated otherwise, a variable cannot, in

one and the same compensation situation, be both compensated and compensating. Moreover, in any compensation situation whatsoever, every variable is either compensated or compensating. We denote by $I=\{(A^+_1, A^-_1),\ldots, (A^+_t, A^-_t)\}$ the set of pairs of permitted compensations. The set of variables which compensate for others equals:

$$I^+ = \{i\in\{1,\ldots,n\} \;/\; \exists(A^+,A^-)\in I \text{ such that } i\in A^+\}$$

The set of variables which are compensated by others equals:

$$I^- = \{ i\in\{1,\ldots,n\} \;/\; \exists(A^+,A^-)\in I \text{ such that } i\in A^-\}$$

For $i\in I^-$, we write $I^+(i) = \{ j\in I^- \;/\; \exists(A^+,A^-)\in I$ such that $i\in A^-$ and $j\in A^+ \}$ and $I^-(i) = \{ j\in I^+ \;/\; \exists(A^+,A^-)\in I$ such that $i\in A^-$ and $j\in A^+ \}$.

All the compensation situations are modeled in a unique global utility function $U(R_1,\ldots,R_n) = \Sigma_{i\in\{1,\ldots,n\}} V_i(R_i)$. To specify $U(R)$ in the compensation situations of I, it suffices to determine the positive part of V_j for all $j\in I^+$, and the negative part of V_i for all $i\in I^-$. Nevertheless, since we are in a fuzzy logic context, compensation will necessarily be allowed at least a little outside of the compensation situations such as they have been defined by the expert. This shows that $U(R)$ will have to be defined for all R, and hence that the positive and negative parts of V_i must be specified for all $i\in\{1,\ldots,n\}$. However, the expert will not be capable of actually specifying the compensation outside of the compensation situations that he has defined. We will therefore ask for the minimum information possible in these zones.

For each $i\in N$, we want to know to which case out of those described above (in regard to the typical compensation behaviors) the variable R_i corresponds. Let us recall that these cases describe the behavior of

the level curves for the bad values of the variable R_i. For $i \in I^-$, we pose the question:

Out of the following three possible behaviors, which one corresponds to R_i?

We have the following three possible answers R1 to R3:

<u>R1:</u> There exists a value of R_i above (if $\varepsilon_i=1$)/below (if $\varepsilon_i=-1$) which no more compensation is possible regardless of the value along the other variables.

<u>R2:</u> Regardless of the value R_i (even very bad), we compensate entirely in respect of sufficiently good values of the other variables.

<u>R3:</u> There exists a value of R_i above (if $\varepsilon_i=1$)/below (if $\varepsilon_i=-1$) which we no longer compensate entirely regardless of the value along the other variables. Moreover, regardless of the value of R_i (even very bad), we always compensate at least a little in respect of sufficiently good values of the other variables.

We denote by $\Re_1$ the set of values of i belonging to the case R1. We denote by $\Re_2$ the set of values of i belonging to the case R2. We denote by $\Re_3$ the set of values of i belonging to the case R3.

As already mentioned previously, the utility functions are characterized by the fact that they have a minimum value, a maximum value, and that they pass through the zero level. We may therefore limit ourselves to determining these three characteristic points.

When $i \in \Re_1$, the shape of the utility function is a little more complex. It comprises two negative levels $V_{i,*}$ and $V_{i,**}$. In order to be able to give a rigorous definition of $R_{i,**}$ referring to $V_{i,**}$, we confine ourselves to compensation situations I such as they are defined by the expert. We recall that $V_{i,**}$ satisfies

the relation $V_{i,**} + \sum_{j\in N\setminus i} V_j^* \leq -1$. Now, the expert is concerned with the compensation situations such as he has defined them, in which the variable R_i appears. This involves pairs $(A^+,A^-)\in I$ such that $i\in A^-$. It is therefore necessary to satisfy the relation $V_{i,**} + \Sigma_{j\in A+} V_j^* \leq -1$ for all $(A^+,A^-)\in I$ such that $i\in A^-$. Moreover, in order to be able to define $R_{i,**1}$ very accurately, we impose the need for the relation $V_{i,**} + \Sigma_{j\in A+} V_j^* \leq -1$ to be satisfied with an equality for a pair $(A^+,A^-)\in I$ with $i\in A^-$. Unfortunately, it is not possible to satisfy this constraint and the relation $V_{i,**} + \sum_{j\in N\setminus i} V_j^* \leq -1$ at the same time. To resolve this difficulty, we introduce two negative levels $V_{i,*}$ and $V_{i,**}$. Let $R_{i,*}$ be the value of the variable R_i corresponding to the utility $V_{i,*}$: $V_{i,*}=V_i(R_{i,*})$. The value $V_{i,**}$ satisfies the relation $V_{i,**} + \sum_{j\in N\setminus i} V_j^* = -1$, so that $R_{i,**}$ has a clear and precise definition outside of the compensation framework. The value $V_{i,*}$ satisfies the relation $V_{i,**} + \Sigma_{j\in A+} V_j^* \leq -1$ for all $(A^+,A^-)\in I$ such that $i\in A^-$ (one of these inequalities corresponding to an equality), so that $R_{i,*}$ has a clear and precise definition referring to the case of the compensation situations defined by the expert. The curve of figure 14 gives the profile of the utility functions when they are increasing ($\varepsilon_i=1$).

In the case where i belongs to $\Re_2$ or $\Re_3$, there is no need to introduce $R_{i,*}$ and $V_{i,*}$. Nevertheless, in what follows, we will refer to the values $R_{i,*}$ and $V_{i,*}$ when we are in the compensation situations of I. By uniformity, we also define $R_{i,*}$ and $V_{i,*}$ in the case where i belongs to $\Re_2$ or $\Re_3$. In this case, we write $R_{i,*} = R_{i,**}$ and $V_{i,*} = R_{i,**}$.

We have for all $i\in\Re_1$:

$$\forall\ (A^+,A^-)\in I \text{ such that } i\in A^-, \text{ we have } V_{i,*} + \Sigma_{j\in A+} V_j^* \leq -1$$

We have for all $i\in\Re_2$:

$$\exists (A^+,A^-)\in I \text{ such that } i\in A^-, \text{ we have } 0 < V_{i,*} + \Sigma_{j\in A+} V_j^*$$

The case R3 is the complement of the union of the first two cases. We have for all $i \in \Re_3$:

$$\exists (A^+, A^-) \in I \text{ such that } i \in A^-, \text{ we have } V_{i,*} + \Sigma_{j \in A+} V_j^* > -1$$

$$\forall (A^+, A^-) \in I \text{ such that } i \in A^-, \text{ we have } V_{i,*} + \Sigma_{j \in A+} V_j^* \leq 0$$

We will now examine the case of the reply R1 alone. We assume here that for all i, we have $i \in \Re_1$.

Firstly, we will examine the utility functions determined at the extremities. In this part we limit ourselves to the determination, for all i, of the characteristic points s_i, R_i^*, $R_{i,*}$, $R_{i,**}$ and their utility. We assume that the utility functions are either simply affine or given by the expert between these points. The expert will be posed questions relating to the two level curves "we compensate entirely" and "we do not compensate at all" concerning these characteristic points only. We assume that the expert will be capable of explaining alone the remainder of the utility functions, that is to say their behavior between the characteristic points.

We must therefore determine for every i, the seven notable values s_i, R_i^*, $R_{i,*}$, $R_{i,**}$, V_i^*, $V_{i,*}$ and $V_{i,**}$. To do this, we shall base ourselves on the consequences of these values, that is to say the possibility of the alternative deriving from U. Nevertheless, it will not be at all easy for an expert to be able to provide, from the value of the variables, directly the degree of possibility regarding the alternative deriving from U. On the other hand, he will be capable of saying whether the alternative C_m is entirely possible or whether it is impossible. Stated otherwise, the determination of the parameters may be done based on notable points of the 0 level curve (the alternative C_m is entirely possible) and -1 level curve (the alternative C_m is impossible) of U.

We will here list the particular points of the two level curves, 0 and -1, of U that make it possible to enable the parameters to be determined.

Let $(A^+,A^-)\in I$, $i\in A^-$ and $A\subset A^+$. We define the vector of variables $R_{i,A}{}^-(R_i)$ by: $(R_{i,A}{}^-(R_i))_i=R_i$, $(R_{i,A}{}^-(R_i))_k=s_k$ for $k\in A^-\backslash i$ and for $k\in A^+\backslash A$, and $(R_{i,A}{}^-(R_i))_k=R_k{}^*$ for $k\in A$. According to this definition, for R_i lying between $R_{i,*}$ and s_i, the variables of A^+ are good while the variables of A^- are bad. Hence, the variables in A^+ do indeed compensate for the variables of A^-. We have the compensation such as it is desired by the expert. We have:

$$U(\ R_{i,A}{}^-(R_i)\ ) = V_i(R_i) + \Sigma_{k\in A}\ V_k{}^*$$

For $R_i=s_i$, we have $U(\ R_{i,A}{}^-(s_i)) = \Sigma_{k\in A}\ V_k{}^*\geq 0$. For $R_i= R_{i,*}$, as the compensation is of the type R1, we have:

$$U(\ R_{i,A}{}^-(R_{i,*})\ ) = V_{i,*} + \Sigma_{k\in A}\ V_k{}^* \leq V_{i,*} + \Sigma_{k\in A+}\ V_k{}^* \leq -1$$

Hence, by continuity of the utility function V_i, there exists R_i lying between $R_{i,*}$ and s_i such that $U(R_{i,A}{}^-(R_i))=0$. This is the smallest (if $\varepsilon_i=1$)/largest (if $\varepsilon_i=-1$) value of R_i for which the alternative C_m deserves entirely to be allocated to the point $R_{i,A}{}^-(R_i)$. Moreover, there exists R_i lying between $R_{i,*}$ and s_i such that $U(R_{i,A}{}^-(R_i))=-1$. This is the largest (if $\varepsilon_i=1$)/smallest (if $\varepsilon_i=-1$) value of R_i for which the alternative C_m is completely impossible for the point $R_{i,A}{}^-(R_i)$.

An interesting particular case occurs when $A=\varnothing$. In this case, we have: $U(R_{i,\varnothing}{}^-(R_i)) = V_i(R_i)$. Therefore $U(R_{i,\varnothing}{}^-(s_i)) = 0$.

Within the framework of the determination of the parameters, and to avoid the effects of thresholds, we must determine U, not only for all the possible values of the variables within the framework of compensation, but for all the possible values of the variables even

outside of the framework of compensation. Nevertheless, insofar as possible, we will have to ask the expert only questions relating to values situated within the framework of compensation.

We determine the utility functions as and when required. While the algorithm is running, we denote by D^+ the set of variables R_j for which the positive part of the utility function V_j is determined, and by D^- the set of variables R_i for which the positive part of the utility function V_i is determined. At the start of the algorithm, we have $D^+=\emptyset$ and $D^-=\emptyset$.

We now give the details of the algorithm. The steps are numbered by a label always beginning with C1-R1. The title C1 signifies that we are in case 1 (that is to say we determine the utility functions only of the extremities) while R1 signifies that all the variables are assumed to belong to the compensation framework R1.

The method of the invention is composed of the following steps:

C1-R1-1 - Definition of the reference thresholds $s_1,\dots,s_n$: The reference thresholds $s_1,\dots,s_n$ correspond to the level which makes it possible to compensate entirely, but only just, as regards the variables $R_1,\dots,R_n$. To determine s_i, the expert answers the following question:

> **Q1 [C1-R1]: What is the value that makes it possible to allocate the alternative C_m entirely, but only just, according to the variable R_i alone?**

Another possible question would be:

> **Q1′ [C1-R1]: What is the value that makes it possible to allocate the alternative C_m entirely, but only just, as regards the**

variable R_i, if the other thresholds are fixed at the same level of satisfaction?

We must have:

(1-[C1-R1]) $V_i(s_i)=0$

<u>C1-R1-2 - Definition of the values R_j^* for $j \in I^+$:</u> Onward of (if $\varepsilon_j=1$)/below (if $\varepsilon_j=-1$) R_j^*, the utility function V_j remains stuck at the value V_j^* and makes no further progress. To determine R_j^*, the expert answers the following question:

Q2 [C1-R1]: Onward of (if $\varepsilon_j=1$)/below (if $\varepsilon_j=-1$) what value of R_j does this variable no longer compensate further for the variables of $I^-(j)$ in the allocation of the alternative C_m?

We must have $\varepsilon_j \times R_{j,*} \geq \varepsilon_j \times R_{j,\blacklozenge}$, or $R_{j,\blacklozenge}$ is defined above (in regard to the methodology of explanation of fuzziness in a noncompensatory condition).

<u>C1-R1-3 - Definition of the values $R_{k,*}$ for $k \in I^-$:</u> Within the framework of a compensation of the type R1, $R_{k,*}$ is the value of R_k below which no compensation will be possible, regardless of the value along the other variables (in a compensation situation of I). This implies that we should not compensate at all when $R_k=R_{k,*}$ for the best possible values along the other variables, within the framework of compensation. Here we shall recall the condition given previously for R1:

(2-[C1-R1]) $\forall (A^+,A^-) \in I$ **with** $k \in A^-$, $V_{k,*} + \Sigma_{j \in A+} V_j^* \leq -1$.

To determine $R_{k,*}$, the expert answers the following question:

Q3 [C1-R1]: Below (if $\varepsilon_k=1$)/onward of (if $\varepsilon_k=-1$) what value of R_k do you want to no longer compensate at all, regardless of the value along the other variables (with values corresponding to a compensation situation)?

Another possible question would be:

Q3′ [C1-R1]: Up to (if $\varepsilon_k=1$)/onward of (if $\varepsilon_k=-1$) what value of R_k does the alternative C_m no longer deserve to be envisaged (with values corresponding to a compensation situation)?

We must have $\varepsilon_k \times R_k^* \le \varepsilon_k \times R_k^\blacklozenge$, where $R_k^\blacklozenge$ is defined hereinabove (in regard to the methodology of explanation of fuzziness in a noncompensatory condition). According to this definition, R_k^* is the limit value of the compensation. Hence, in (2-[C1-R1]), we must have equality for a pair (A^+,A^-).

<u>**C1-R1-4 - Determination of the utility function V_k between $R_{k,*}$ and s_k for $k\in A^-$:**</u> At this stage, the value of $V_{k,*}$ is still not specified. On the other hand, according to equation (1-[C1-R1]), we know that V_k corresponds to a ratio scale. The objective of this phase is to be specify V_k in the guise of ratio scale, that is to say to determine $\lambda_k(R_k)=V_k(R_k)/\ V_{k,*}$. This number lies between 0 and 1 for R_k lying between $R_{k,*}$ and s_k. To determine $\lambda_k(R_k)$, we use a procedure arising out of measure theory. The invention provides for the MACBETH methodology (cf. C. Bana e Costa & J.C. Vansnick, *Applications of the MACBETH approach in the framework of an additive aggregation model,* Journal of Multicriteria Decision Analysis, No. 6, pp. 107-114, 1997). Nevertheless, any other methodology making it possible to determine utility functions is also suitable. In the simplest case, λ_k is simply affine: $\lambda_k(R_k)=(s_k-R_k)/(s_k-R_{k,*})$.

<u>**C1-R1-5 - Determination of the utility function V_j between s_j and R_j^* for $j\in A^+$:**</u> We proceed exactly as in the previous phase. We therefore determine with the help of a procedure arising out of measure theory the value of $\lambda_j(R_j)=V_j(R_j)/\ V_j^*$. This number lies between 0 and 1 for R_j lying between s_j and R_j^*. In the simplest case, λ_j is simply affine: $\lambda_j(R_j)=(R_j-s_j)/(R_j^*-s_j)$.

C1-R1-6 - Determination of $i \in I^- \backslash D^-$ of reference: The questionnaire will be based on a particular variable from $I^- \backslash D^-$. The objective of this step is to determine this index. It is the most important variable among those which are compensated (in $I^- \backslash D^-$). The compensation that we describe generally pertains chiefly to one variable. This is the variable that we are seeking. To determine $i \in I^-$, the expert answers the following question:

> **Q4 [C1-R1] : Which out of the remaining variables that are compensated (that is to say $I^- \backslash D^-$) is the most important, the one that the questionnaire will be based on?**

C1-R1-7 - Determination of $V_{i,*}$ and of V_j^* for all $j \in I^+(i) \backslash D^+$: Let $j \in I^+(i) \backslash D^+$. Let $(A^+, A^-) \in I$ such that $i \in A^-$ and $j \in A^+$. We will use a singular point defined hereinabove (in regard to the particular points on the two level curves). We consider the vector of variables $R_{i,j}^-(R_i)$ defined previously with the sets A^+ and A^-. $R_{i,j}^-(R_i)$ corresponds $R_{i,A}^-(R_i)$ when $A=\{j\}$. The vector of variables $R_{i,j}^-(R_i)$ is defined by: $(R_{i,j}^-(R_i))_i=R_i$, $(R_{i,j}^-(R_i))_j=R_j^*$, and $(R_{i,j}^-(R_i))_k=s_k$ for $k \in N \backslash \{i,j\}$. We therefore see that $R_{i,j}^-(R_i)$ is independent of A^+ and A^-. We saw above (again with regard to the particular points), that there exists R_i lying between $R_{i,*}$ and s_i such that $U(R_{i,j}^-(R_i))=0$. This value is denoted $R_{i,j}^0$. It is independent of A^+ and A^-. It is the smallest (if $\varepsilon_i=1$)/largest (if $\varepsilon_i=-1$) value of R_i such that the alternative C_m deserves entirely to be allocated to the point $R_{i,j}^-(R_i)$. To determine $R_{i,j}^0$, the expert answers the following question:

> **Q5[C1-R1] : For R_k fixed at s_k for $k \neq i,j$, onward of (if $\varepsilon_i=1$)/up to (if $\varepsilon_i=-1$) what value of R_i do you think that the variable $R_j=R_j^*$ compensates entirely for R_i?**

As $R_{i,j}^-(R_{i,j}^0))=0$, we must have:

$$V_j^* + \lambda_i(R_{i,j}^0)\ V_{i,*} = 0 \qquad \textbf{(3-[C1-R1])}$$

We write, according to equation (3-[C1-R1]), V_j^* as a function of $V_{i,*}$. By placing this in (2-[C1-R1]), we obtain

$$(1-\Sigma_{j\in A+\backslash D+}\ \lambda_i(R_{i,j}^0))\ V_{i,*} + \Sigma_{j\in A+\cap D+}\ V_j^* \leq -1\ .$$

We cannot be in the compensation case R1 if $1-\Sigma_{j\in A+\backslash D+}\ \lambda_i(R_{i,j}^0)\leq 0$. We therefore assume here that $1-\Sigma_{j\in A+\backslash D+}\ \lambda_i(R_{i,j}^0)>0$. In the converse case, the information provided is inconsistent. $V_{i,*}$ therefore satisfies:

$$\mathbf{V_{i,*} \leq -(1+\Sigma_{j\in A+\cap D+}\ V_j^*)\ /\ (1-\Sigma_{j\in A+}\ \lambda_i(R_{i,j}^0))}$$

This relation must be satisfied for all $(A^+,A^-)\in I$ such that $i\in A^-$. We therefore obtain:

$$\mathbf{V_{i,*} \leq \wedge_{(A+,A-)\in I\ /\ i\in A-}\ -(1+\Sigma_{j\in A+\cap D+}\ V_j^*)\ /\ (1-\Sigma_{j\in A+}\ \lambda_i(R_{i,j}^0))}$$

where the operator $\wedge$ designates the minimum.

In order for $R_{i,*}$ to actually correspond to the largest (if $\varepsilon_i=1$)/smallest (if $\varepsilon_i=-1$) value of R_i which we no longer compensate at all, the previous relation must be considered with an equality. This signifies that the relation $V_{i,*} + \Sigma_{j\in A+}\ V_j^* \leq -1$ will be satisfied with an equality for a pair (A^+,A^-). Hence:

(4-[C1-R1]) $\mathbf{V_{i,*}=\wedge_{(A+,A-)\in I/i\in A-}\ -(1+\Sigma_{j\in A+\cap D+}\ V_j^*)/(1-\Sigma_{j\in A+}\ \lambda_i(R_{i,j}^0))}$

From this we deduce with (3-[C1-R1]) the expression for V_j^* for all $j\in I^+(i)\backslash D^+$:

(5-[C1-R1]) $\mathbf{V_j^*=-\lambda_i(R_{i,j}^0)\times\wedge_{(A+,A-)\in I/i\in A-}\ -(1+\Sigma_{j\in A+\cap D+}\ V_j^*)/(1-\Sigma_{j\in A+}\ \lambda_i(R_{i,j}^0))}$

We add the variable i to the set D^- and the set $I^+(i)\backslash D^+$ to D^+.

C1-R1-8 - Determination of $V_{k,*}$ for $k\in I^-\backslash D^-$ such that $I^+(k)\subset D^+$: For all $k\in I^-\backslash D^-$ such that $I^+(i)\subset D^+$, we have according to (2-[C1-R1]), regardless of $(A^+,A^-)\in I$ with $k\in A^-$, we have:

$$V_{k,*} \leq -1 - \Sigma_{j\in A+} V_j^*$$

hence:

$$V_{k,*} \leq \wedge_{(A+,A-)\in I/k\in A-} \; -1-\Sigma_{j\in A+} V_j^*$$

As previously, the previous relation must be considered with an equality. We therefore have $k\in I^-\backslash D^-$ such that $I^+(i)\subset D^+$:

(6-[C1-R1]) $\quad V_{k,*} = \wedge_{(A+,A-)\in I/k\in A-} \; -1-\Sigma_{j\in A+} V_j^*$

We add the variables $k\in I^-\backslash D^-$ such that $I^+(k)\subset D^+$ to the set D^-. If $D^-\neq I^-$, we return to step C1-R1-6. If $D^- = I^-$, then necessarily we have $D^+ = I^+$.

C1-R1-9 - Determination of V_j^* for a $j\notin I^+$: The variable R_j for $j\notin I^+$ is never presumed to compensate for other variables, according to the possible compensations fixed by the user. Nevertheless, to avoid the threshold effects, it is indeed necessary for all the compensations to be defined. To determine R_j^*, we ask the expert a question much like the question Q2[C1-R1] :

> **Q6 [C1-R1]: Up to (if ε_j=1)/onward of (if ε_j=-1) what value of R_j can this variable no longer compensate any further for the other variables in the allocation of the alternative C_m?**

We will not ask the expert a question of the type Q5[C1-R1] to determine V_j^*. We want to determine a value of V_j^* without asking a question, less it be empirical. The expert will not be capable of reasoning in these zones, and hence of answering a precise question of the type Q5[C1-R1]. The exact value of the compensation in these zones is of lesser importance.

As this variable is not presumed to compensate for others, the manner in which it will compensate for the others will necessarily be of small amplitude, that is to say it will compensate less than the variables that are presumed to compensate:

$$V_j^* < \wedge_{k\in I+} V_k^* \;.$$

We can for example:

$$V_j^* = \wedge_{k \in I+} V_k^*/2 \ .$$

<u>C1-R1-10 - Determination of V_k beyond $R_{k,*}$ for $k \in I^-$:</u> To determine $R_{k,**}$, we ask the expert a question much like the question Q3 [C1-R1]:

> **Q7 [C1-R1]: Below (if ε_k=1)/onward of (if ε_k=−1) what value of R_k do you no longer want to compensate at all, regardless of the value along the other variables (with values outside of the framework of compensation)?**

To determine the extent to which the compensation should occur in the permitted compensation zones, it suffices to know V_k between s_k and R_k*. The values of $R_{k,**}$ and $V_{k,**}$ have no influence in the zones where compensation is permitted. According to the definition of $R_{k,**}$, $V_{k,**}$ must satisfy the relation:

$$V_{k,**} + \Sigma_{j \neq k} V_j^* = -1$$

We therefore obtain:

(7-[C1-R1]) $\quad V_{k,**} = -1 - \Sigma_{j \neq k} V_j^*$

<u>C1-R1-11 - Determination of $R_{k,*}$ and of $V_{k,*}$ for $k \notin I^-$:</u> For $k \notin I^-$, the variable R_k is never presumed to be compensated by other variables. To determine $R_{k,*}$ and $V_{k,*}$, we proceed as in the previous step. To determine $R_{k,*}$, we ask the expert a question much like the question Q3 [C1-R1] :

> **Q8 [C1-R1]: Below (if ε_k=1)/onward of (if ε_k=−1) what value of R_k do you no longer want to compensate at all, regardless of the value along the other variables?**

According to the definition of $R_{k,*}$, we obtain, as previously, for $V_{k,*}$ the following expression:

(8-[C1-R1]) $\quad V_{k,*} = -1 - \Sigma_{j \neq k} V_j^*$

At the end of steps C1-R1-7 and C1-R1-8, the construction certifies that the condition $V_{i,*} + \Sigma_{j \in A+} V_j^* \leq -1$ is indeed satisfied for all $(A^+, A^-) \in I$ such that $i \in A^-$.

Once the parameters have been calculated, we can validate them with reference to the notable points. Let: $(A^+, A^-) \in I$. We then examine the various cases of vectors of variables.

- Vector of variables $R_{k,\varnothing}^-$ for $k \in A^-$. According to the above explanations relating to the particular points on the two level curves; there exists a value $R_{k,\varnothing}^{-1}$ of the variable R_k such that $U(R_{k,\varnothing}^-(R_{k,\varnothing}^{-1})) = -1$. As $U(R_{k,\varnothing}^-(R_{k,\varnothing}^{-1})) = V_k(R_k)$, we have:
  $R_{k,\varnothing}^{-1} = V_k^{-1}(-1)$

$R_{k,\varnothing}^{-1}$ is interpreted in the following manner:

Up to (if ε_k=1)/onward of (if ε_k=−1) the value $R_{k,\varnothing}^{-1}$ of the variable R_k, the variables R_p=s_p for $p \in A^+$ no longer compensate at all for R_k and R_p=s_p for $p \in A^- \backslash \{i\}$.

This may also be stated as follows:

Up to (if ε_k=1)/onward of (if ε_k=−1) the value $R_{k,\varnothing}^{-1}$ of the variable R_k, the alternative C_m is impossible when the other variables are at the level of the threshold s_p.

If the expert is not in agreement with the value of $R_{k,\varnothing}^{-1}$, we set $\lambda_k(R_{k,\varnothing}^{-1}) = -1/V_{k,*}$ for the value $R_{k,\varnothing}^{-1}$ that the expert thinks is correct.

- Vector of variables $R_{k,j}^-$ for $k \in A^- \backslash \{i\}$ and $j \in A^+$. According to the same explanations as regards the particular points, there exists a value $R_{k,j}^{-1}$ of the

variable R_k such that $U(R_{k,j}^-(R_{k,j}^{-1})) = -1$. As $U(R_{k,j}^-(R_{k,j}^{-1})) = V_k(R_k) + V_j^*$, we have:

$R_{k,j}^{-1} = V_k^{-1}(-1- V_j^*)$

$R_{k,j}^{-1}$ is interpreted in the following manner:

Up to (if ε_k=1)/onward of (if ε_k=-1) the value $R_{k,j}^{-1}$ of the variable R_k, the variables $R_k = R_k^*$ and $R_p = s_p$ for $p \in A^+ \setminus \{j\}$ no longer compensate at all for R_k and $R_p = s_p$ for $p \in A^- \setminus \{i\}$.

This may also be stated in the following manner:

Up to (if ε_k=1)/onward of (if ε_k=-1) the value $R_{k,j}^{-1}$ of the variable R_k, the alternative C_m is impossible when the other variables are at the level of the threshold s_p, except the variable $R_j = R_j^*$.

If the expert is not in agreement with the value of $R_{k,j}^{-1}$, we set $\lambda_k(R_{k,j}^{-1}) = -(1+ V_j^*)/ V_{k,*}$ for the value $R_{k,j}^{-1}$ that the expert thinks is correct.

- Vector of variables $R_{k,j}^-$ for $k \in A^- \setminus \{i\}$ and $j \in A^+$. According to the same explanations as regards the particular points, there exists one value $R_{k,j}^0$ of the variable R_k such that $U(R_{k,j}^-(R_{k,j}^0)) = 0$. We have:

  $R_{k,j}^0 = V_k^{-1}(- V_k^*)$

$R_{i,j}^0$ is interpreted in the following manner:

Onward of (if ε_k=1)/below (if ε_k=-1) the value $R_{k,j}^{-0}$ of the variable R_k, the variables $R_k = R_k^*$ and $R_p = s_p$ for $p \in A^+ \setminus \{j\}$ compensate entirely for R_k and $R_p = s_p$ for $p \in A^- \setminus \{i\}$.

This may also be stated in the following manner:

Onward of (if ε_k=1)/below (if ε_k=-1) the value $R_{k,j}^{-1}$ of the variable R_k, the alternative C_m is entirely possible when the other variables are at the level of the threshold s_p, except the variable $R_j = R_j^*$.

If the expert is not in agreement with the value of $R_{k,j}^{-0}$, we set $\lambda_k(R_{k,j}^{0}) = -V_j^{*}/V_{k,*}$ for the value $R_{k,j}^{-0}$ that the expert thinks is correct.

We will now examine the utility functions determined at every point for I^{+} and I^{-}. The hypothesis made above in regard to the utility functions determined at the extremities is that the expert is capable of explaining the utility functions between the extreme values which themselves are determined by the methodology. Thus, in accordance with steps C1-R1-4 and C1-R1-5 described hereinabove, the expert must be capable of providing $\lambda_i(R_i)$ for all R_i. This is arguable given the implications that the values of λ_i have on the level curves. It is classical to consider the utility functions as piecewise affine functions. It may then seem desirable that the methodology should make it possible to determine the levels $V_i(R_i)$ of each point R_i by delimiting two affine parts.

In this section, we assume that the expert is not capable of determining $\lambda_i(R_i)$ between the two extremities for $i\in I^{-}$ or $i\in I^{+}$.

For $j\in I^{+}$, we assume that the expert is capable of providing the relevant intermediate points between s_j and R_j^{*}. We write $R_j^{1},\ldots,R_j^{pj}$ for these p_j points, and $V_j^{k}=V_j(R_j^{k})$ for $k\in\{1,\ldots,p_j\}$, as represented in figure 15.

We write $R_j^{0} = s_j$, $V_j^{0} = 0$, $R_j^{pj+1}= R_j^{*}$ and $V_j^{pj+1}= V_j^{*}$. For $k\in\{0, \ldots, p_j\}$ and $R_j\in[R_j^{k}, R_j^{k+1}]$, we have $V_j(R_j)= V_j^{k}+(V_j^{k+1} - V_j^{k})\lambda_j^{k}(R_j)$ where $\lambda_j^{k}(R_j) = (R_j - R_j^{k})/(R_j^{k+1} - R_j^{k})$.

For $i\in I^{-}$, we assume that the expert is capable of providing the relevant intermediate points between $R_{i,*}$ and s_i. We write $R_i^{-1},\ldots,R_i^{-p'i}$ for these p'_i points, and $V_i^{k}=V_i(R_i^{k})$ for $k\in\{-p'_i,\ldots,-1\}$. We write $R_i^{0}= s_i$, $V_i^{0}=0$, $R_i^{-p'i-1}=R_{i,*}$ and $V_i^{-p'i-1}=V_{i,*}$. For all $k\in\{-p'_i-1,\ldots,-1\}$ and all

$R_i \in [R_i^k, R_i^{k+1}]$, we have $V_i(R_i) = V_i^k + (V_i^{k+1} - V_i^k)\lambda_i^k(R_i)$ where $\lambda_i^k(R_i) = (R_i - R_i^k)/(R_i^{k+1} - R_i^k)$. The utility function is affine in each segment $[R_i^k, R_i^{k+1}]$, as represented in the curve of figure 16. Here we seek, according to the 0 and -1 level curves, to determine $V_i^{-1}, \ldots, V_i^{-p'i}$, $V_{i,*}$ for $i \in I^-$, and $V_j^*, V_j^1, \ldots, V_j^{pj}$ for $j \in I^+$.

We will list the particular points of the two 0 and -1 level curves of U that make it possible to enable the parameters to be determined. Let $(A^+, A^-) \in I$.

Let K be a set of indices k_j for $j \in A^+$, with $k_j \in \{0, \ldots, p_j+1\}$. Let $i \in A^-$ and $K = \{k_j\}_{j \in A+}$. We define the vector of variables $R_i^{K-}(R_i)$ by: $(R_i^{K-}(R_i))_i = R_i$, $(R_i^{K-}(R_i))_q = s_q$ for $q \in A^- \backslash i$, and $(R_i^{K-}(R_i))_q = R_q^{kq}$ for $q \in A^+$. We have:

$$U(R_i^{K-}(R_i)) = V_i(R_i) + \Sigma_{j \in A+} V_j^{kj}$$

For $R_i = s_i$, it follows that:

$$U(R_i^{K-}(s_i)) = \Sigma_{j \in A+} V_j^{kj} \geq 0.$$

For $R_i = R_{i,*}$, as the compensation is of the type R1, we have:

$$U(R_i^{K-}(R_{i,*})) = V_{i,*} + \Sigma_{j \in A+} V_j^{kj} \leq V_{i,*} + \Sigma_{k \in A+} V_k^* \leq -1$$

Hence, by continuity, there exists R_i lying between s_i and $R_{i,*}$ such that $U(R_i^{K-}(R_i)) = 0$. We write $R_i^{K,0}$ for this point. This is the smallest (if $\varepsilon_i = 1$)/largest (if $\varepsilon_i = -1$) value of R_i for which the alternative C_m deserves entirely to be allocated to the point $R_i^{K-}(R_i)$. We have:

$$V_i(R_i^{K,0}) + \Sigma_{j \in A+} V_j^{kj} = 0$$

Moreover, there exists R_i lying between s_i and $R_{i,*}$ such that $U(R_i^{K-}(R_i)) = -1$. We write $R_i^{K,-1}$ for this point. This is the largest (if $\varepsilon_i = 1$)/smallest (if $\varepsilon_i = -1$) value of R_i for which the alternative C_m is completely impossible for the point $R_{i,A}^-(R_i)$. We have:

$$V_i(R_i^{K,-1}) + \Sigma_{j \in A+} V_j^{kj} = -1$$

Let $k_i\in\{-p'_i-1,\ldots,-1\}$, and consider a set of indices k_j for $j\in A^+\setminus\{j\}$, with $k_j\in\{0,\ldots,p_j+1\}$. We now define another notable point. Let $i\in A^-$, $j\in A^+$ and $K=\{k_j\}_{j\in i\cup A+\setminus j}$. We define the vector of variables $R_{i,j}^{K+}(R_j)$ by: $(R_{i,j}^{K+}(R_j))_j=R_j$, $(R_{i,j}^{K+}(R_j))_q=s_q$ for $q\in A^-\setminus i$, and $(R_{i,j}^{K+}(R_j))_q=R_q^{kq}$ for $q\in i\cup A^+\setminus j$. We have:

$$U(R_{i,j}^{K+}(R_j)) = V_j(R_j) + V_i^{ki} + \Sigma_{q\in A+\setminus j} V_q^{kq}$$

In contradiction to the previous notable point, it is not definite that when R_j describes the interval lying between s_j and R_j^* then $U(R_{i,j}^{K+}(R_j))$ cuts one of the two 0 or -1 level curves. We will study the conditions under which there is intersection. To do this, let K^0 be the index set defined on A^+ by $(K^0)_q=k_q$ if $q\in A^+\setminus j$ and $(K^0)_j=0$. We write $R_i^{K0,0}$ and $R_i^{K0,-1}$ for the notable points of the vector of variables $R_i^{K0,-}$ defined previously on the basis of K^0. Let K^* be the index set defined on A^+ by $(K^*)_q=k_q$ if $q\in A^+\setminus j$ and $(K^*)_j= p_j+1$. We write $R_i^{K*,0}$ and $R_i^{K*,-1}$ for the notable points of the vector of variables $R_i^{K*,-}$ defined previously on the basis of K^*. Finally, consider the vector of variables $R_{i,j}^K(R_i,R_j)$ by: $(R_{i,j}^K(R_i,R_j))_i=R_i$, $(R_{i,j}^K(R_i,R_j))_j=R_j$, $(R_{i,j}^K(R_i,R_j))_q=s_q$ for $q\in A^-\setminus i$, and $(R_{i,j}^K(R_i,R_j))_q=R_q^{kq}$ for $q\in A^+\setminus j$. We have:

$$U(R_{i,j}^K(R_i,R_j)) = V_i(R_i) + V_j(R_j) + \Sigma_{q\in A+\setminus j} V_q^{kq}$$

We have the following cases:

- If $\varepsilon_i \times R_i < \varepsilon_i \times R_i^{K*,-}$, then for all $R_j\in[R_j^*,s_j]$, we have, as V_i is increasing (if $\varepsilon_i=1$)/decreasing (if $\varepsilon_i=-1$) and according to the definition of $R_i^{K*,-}$:

  $$V_i(R_i) < V_i(R_i^{K*,-}) = -1-V_j^*-\Sigma_{q\in A+\setminus j} V_q^{kq}$$

  Hence:

  $$U(R_{i,j}^K(R_i,R_j)) \le V_i(R_i) + V_j^* + \Sigma_{q\in A+\setminus j} V_q^{kq} < -1$$

  From this we deduce that the level curve $U(R_{i,j}^K(R_i,R_j)) = -1$ does not pass through the rectangle $\varepsilon_i \times R_i < \varepsilon_i \times R_i^{K*,-}$, $R_j\in[s_j,R_j^*]$.
- Let $\varepsilon_i \times R_i^{K*,-} \le \varepsilon_i \times R_i \le \varepsilon_i \times R_i^{K0,-}$. Then $V_i(R_i^{K*,-}) \le V_i(R_i) \le V_i(R_i^{K0,-})$. We have:

  $$V_i(R_i) \ge V_i(R_i^{K*,-}) \Leftrightarrow U(R_{i,j}^K(R_i,R_j^*)) \ge -1$$

$$V_i(R_i) \le V_i(R_i^{K0,-}) \Leftrightarrow U(R_{i,j}^K(R_i,s_j)) \le -1$$

From this we deduce that there exists $R_j \in [s_j, R_j^*]$ such that $U(R_{i,j}^K(R_i,R_j)) = -1$.

- Let $\varepsilon_i \times R_i^{K*,0} \le \varepsilon_i \times R_i \le \varepsilon_i \times R_i^{K0,0}$. Then $V_i(R_i^{K*,0}) \le V_i(R_i) \le V_i(R_i^{K0,0})$. We have:

$$V_i(R_i) \ge V_i(R_i^{K*,0}) \Leftrightarrow U(R_{i,j}^K(R_i,R_j^*)) \ge 0$$

$$V_i(R_i) \le V_i(R_i^{K0,0}) \Leftrightarrow U(R_{i,j}^K(R_i,s_j)) \le 0$$

From this we deduce that there exists $R_j \in [s_j, R_j^*]$ such that $U(R_{i,j}^K(R_i,R_j)) = 0$.

- If $\varepsilon_i \times R_i > \varepsilon_i \times R_i^{K0,0}$, then, for all $R_j \in [s_j, R_j^*]$

$$U(R_{i,j}^K(R_i,R_j)) \ge U(R_{i,j}^K(s_i,s_j)) > 0$$

From this we deduce that the level curve $U(R_{i,j}^K(R_i,R_j)) = 0$ does not pass through the rectangle $\varepsilon_i \times R_i > \varepsilon_i \times R_i^{K0,0}$, $R_j \in [s_j, R_j^*]$.

Clearly, we have $\varepsilon_i \times R_i^{K*,-} < \varepsilon_i \times R_i^{K0,0}$ since:

$$\varepsilon_i \times R_i^{K*,-} < \varepsilon_i \times R_i^{K0,0} \Leftrightarrow V_i(R_i^{K*,-}) < V_i(R_i^{K0,0}) \Leftrightarrow -1 - V_j^* - \Sigma_{q \in A+\backslash j} V_q^{kq} < \Sigma_{q \in A+\backslash j} V_q^{kq}$$

This last condition is true since $V_j^* \ge 0$.

On the other hand, it is easy to see that it is certain that for all R_i such that $\varepsilon_i \times R_i^{K*,-} \le \varepsilon_i \times R_i \le \varepsilon_i \times R_i^{K0,0}$, at least one of the two level curves $U(R_{i,j}^K(R_i,R_j)) = 0$ or $U(R_{i,j}^K(R_i,R_j)) = -1$ is attained for an $R_j \in [s_j, R_j^*]$ if and only if the two intervals $[R_i^{K*,-}, R_i^{K0,-}]$ and $[R_i^{K*,0}, R_i^{K0,0}]$ intersect. This latter condition states that there is no hole between the two intervals. This may be written $\varepsilon_i \times R_i^{K*,0} \le \varepsilon_i \times R_i^{K0,-}$. We have:

$$\varepsilon_i \times R_i^{K*,0} \le \varepsilon_i \times R_i^{K0,-} \Leftrightarrow V_i(R_i^{K*,0}) \le V_i(R_i^{K0,-}) \Leftrightarrow -V_j^* - \Sigma_{q \in A+\backslash j} V_q^{kq} \le -1 - \Sigma_{q \in A+\backslash j} V_q^{kq} \Leftrightarrow V_j^* \ge 1$$

Stated otherwise, for all R_i such that $\varepsilon_i \times R_i^{K*,-} \le \varepsilon_i \times R_i \le \varepsilon_i \times R_i^{K0,0}$, at least one of the two level curves $U(R_{i,j}^K(R_i,R_j)) = 0$ or $U(R_{i,j}^K(R_i,R_j)) = -1$ is attained for an $R_j \in [s_j, R_j^*]$ if and only if $V_j^* \ge 1$.

We will now return to $R_{i,j}^{K+}(R_j)$. We want the set of values that can be taken by $U(R_{i,j}^{K+}(R_j))$ when R_j describes $[s_j, R_j^*]$ to cut one of the two level curves 0 or -1 for all $k_i \in \{-p_i-1, \ldots, -1\}$. To do this, according to the foregoing, it is necessary for j to satisfy $V_j^* \geq 1$ and that $\varepsilon_i \times R_i^{K*,-} \leq \varepsilon_i \times R_i^k \leq \varepsilon_i \times R_i^{K0,0}$.

We have:

$$\exists\ R_j \in [s_j, R_j^*] \text{ such that } U(R_{i,j}^{K+}(R_j)) = 0$$

$$\Leftrightarrow \varepsilon_i \times R_i^{K*,0} \leq \varepsilon_i \times R_i^k \leq \varepsilon_i \times R_i^{K0,0} \Leftrightarrow V_i^k \in [\ V_i(R_i^{K*,0})\ ,\ V_i(R_i^{K0,0})\ ]$$

$$\Leftrightarrow V_i^k \in [\ -V_j^* - \Sigma_{q \in A+\backslash j}\ V_q^{kq},\ -\Sigma_{q \in A+\backslash j}\ V_q^{kq}\ ]$$

If $k_q=0$ for all $q \in A^+\backslash j$, then $V_q^{kq}=0$ and hence this gives:

$$\exists\ R_j \in [s_j, R_j^*] \text{ such that } U(R_{i,j}^{K+}(R_j)) = 0 \Leftrightarrow V_i^k \in [\ -V_j^*,\ 0]$$

Hence if k is very small, then we expect V_i^k to be close to 0. In this case, the values k_q which will make it possible to cross the zero level curve will be 0 or in any case small.

Moreover we have:

$$\exists\ R_j \in [s_j, R_j^*] \text{ such that } U(R_{i,j}^{K+}(R_j)) = -1$$

$$\Leftrightarrow \varepsilon_i \times R_i^{K*,-} \leq \varepsilon_i \times R_i^k \leq \varepsilon_i \times R_i^{K0,-} \Leftrightarrow V_i^k \in [\ V_i(R_i^{K*,-})\ ,\ V_i(R_i^{K0,-})]$$

$$\Leftrightarrow V_i^k \in [\ -1 - V_j^* - \Sigma_{q \in A+\backslash j}\ V_q^{kq},\ -1 - \Sigma_{q \in A+\backslash j}\ V_q^{kq}]$$

If $k_q=p_q+1$ for all $q \in A^+\backslash j$, then $V_q^{kq}=V_q^*$ and hence this gives:

$$\exists\ R_j \in [s_j, R_j^*] \text{ such that } U(R_{i,j}^{K+}(R_j)) = 0 \Leftrightarrow V_i^k \in [-1 - \Sigma_{q \in A+}\ V_q^*,\ -1 - \Sigma_{q \in A+\backslash j}\ V_q^*] \Leftrightarrow V_i^k \in [V_{i,*},\ V_{i,*} + V_j^*]$$

Hence if k is large, then we expect V_i^k to be close to $V_{i,*}$. In this case, the values k_q which will make it possible to cross the -1 level curve will be equal to p_q+1 or in any case large.

We proceed in the following manner for the determination of the parameters. The utility functions are determined as and when required. During the running of the algorithm, we denote by D^+ the set of variables R_j for which the positive part of the utility function V_j is determined, and by D^- the set of variables R_i for

which the positive part of the utility function V_i is determined. At the start of the algorithm we have $D^+=\varnothing$ and $D^-=\varnothing$.

We shall now give the details of the algorithm. The steps are numbered by a label always beginning with C12-R1. The title C2 signifies that we are in case 2 (that is to say we determine the utility functions at every point) while R1 signifies that all the variables are assumed to belong to the compensation framework R1.

The process of the invention is composed of the following steps:

C2-R1-1 – Definition of the reference thresholds $s_1, \dots, s_n$: This step is strictly identical to step C1-R1-1.

C2-R1-2 – Definition of the values R_j^* for $j \in I^+$: This step is strictly identical to step C1-R1-2.

C2-R1-3 – Definition of the values $R_{k,*}$ for $k \in I^-$: This step is strictly identical to step C1-R1-2.

C2-R1-4 – Determination of the points $R_k^{-1},\dots,R_k^{p-p'k}$ for $k \in I^-$: We ask the expert to give the relevant intermediate points between s_k and $R_{k,*}$. In general it is not necessary to have many of them. From one to three points are sufficient most of the time.

C2-R1-5 – Determination of the points $R_j^1,\dots,R_j^{pj}$ for $j \in I^+$: We ask the expert to give the relevant intermediate points between R_j^* and s_j. We put $D^+=\varnothing$ and $D^-=\varnothing$ before arriving at the next step.

C2-R1-6 – Determination of $i \in I^- \backslash D^-$ of reference, and characterization of V_j^k for all $j \in I^+(i) \backslash D^+$ and $k \in \{1,\dots,p_j+1\}$: We begin by determining a reference index $i \in I^- \backslash D^-$. This step is strictly identical to step C1-R1-

6. In particular we ask the question Q4[C1-R1] to determine this index i.

Let $j \in I^+(i) \setminus D^+$. Let $k \in \{1, \ldots, p_j+1\}$ and let K be the index set defined on A^+ (for A^+ such that $(A^+, A^-) \in I$, $i \in A^-$ and $j \in A^+$) by $K_q=0$ if $q \in A^+ \setminus j$ and $K_j=k$. $R_i^{K,0}$ does not depend on A^+. We ask the expert to provide the value of $R_i^{K,0}$:

Q4[C2-R1]: For R_q fixed at s_q for $q \neq i,j$, onward of (if $\varepsilon_i=1$)/up to (if $\varepsilon_i=-1$) what value of R_i do you think that the variable $R_j = R_j^k$ compensates entirely for R_i ?

As $U(R_i^{K-}(R_i^{K,0})) = 0$, we must have:

(3-[C2-R1]) $$V_j^k + V_i(R_i^{K,0}) = 0$$

From the reference index i, we determine $R_i^{K,0}$ for all $j \in I^+(i) \setminus D^+$ and all $k \in \{1, \ldots, p_j+1\}$. The equation (3-[C2-R1]) therefore provides a relation satisfied by V_j^k for all $j \in I^+(i) \setminus D^+$ and all $k \in \{1, \ldots, p_j+1\}$.

The variable i is added to the set D^-, and the set $I^+(i) \setminus D^+$ is added to D^+. If $D^- \neq I^-$, we return to step C2-R1-6. If $D^- = I^-$, all the values V_j^k (for all $j \in I^+$ and all $k \in \{1, \ldots, p_j+1\}$) are characterized by relation type (3-[C2-R1]). In the next step we will be concerned with characterizing V_i^k.

<u>C2-R1-7 - Determination of a ranking over I^+:</u> We must fix a reference index $j \in I^+ \setminus D^+$. We want to take j for which V_j^* is a maximum. In contradiction to the previous step, since we already have information we want to avoid asking the expert a question to determine the reference index j. We wish to obtain a kind of ranking among the indices of I^+. The reference indices will be taken in the order established by this ranking.

As we do not yet know the value of V_j^*, we base ourselves on the knowledge of the $R_i^{K,0}$, where i is a

reference index of step C2-R1-6. During step C2-R1-6, the expert was able to provide several reference indices i. We therefore start from the order in which the expert provided the reference indices i. Indices j in $I^+(i)\backslash D_i^+$ (where D_i^+ is the value of D^+ at the moment at which step C2-R1-6 is at the level of the reference index i) for a reference index i will all be preferred to the indices $I^+(i')\backslash D_{i'}^+$ for another reference index i' if the expert gave i before i'. The union of all the sets $I^+(i)\backslash D_i^+$ for all the reference indices i is equal to I^+. It now therefore suffices to explain how to order the various indices out of $I^+(i)\backslash D_i^+$ for a fixed index i. For $j\in I^+$, according to equation (3-[C2-R1]), the value $R_i^{K,0}$ satisfies $V_j^*+V_i(R_i^{K,0})=0$. From this we deduce that the smaller is $V_i(R_i^{K,0})$, the larger is V_j^*, and hence that $R_i^{K,0}$ is small (if $\varepsilon_i=1$)/large (if $\varepsilon_i=-1$). We therefore rank the indices $j\in I^+(i)\backslash D_i^+$ in increasing (if $\varepsilon_i=1$)/decreasing (if $\varepsilon_i=-1$) order of $R_i^{K,0}$.

We again put $D^+=\varnothing$ and $D^-=\varnothing$ before going to the next step.

C2-R1-8 - Determination of reference $j\in I^+\backslash D^+$ and characterization of V_i^k for all $i\in I^-(j)\backslash D^-$ and $k\in\{-p'_i-1,\ldots,-1\}$: The reference index $j\in I^+\backslash D^+$ considered is the smallest index belonging to $I^+\backslash D^+$ in the order defined in the previous step. If $I^-(j)\backslash D^-=\varnothing$, we add j to D^+ and we again determine a reference index $j\in I^+\backslash D^+$.

Let $i\in I^-(j)\backslash D^-$. Let $k\in\{-p'_i-1,\ldots,-1\}$. Let $(A^+,A^-)\in I$ with $i\in A^-$ and $j\in A^+$.

We must determine the values of the indices k_q for $q\in A^+\backslash j$ for which the set of values that can be taken by $U(R_{i,j}^{K+}(R_j))$ when R_j describes $[R_j^*,s_j]$ cuts one of the 0 or -1 level curves. We make use of what was established hereinabove in regard to the particular points on the two 0 and -1 level curves of U.

The determination of the indices $\{k_q\}_{q\in A+\backslash j}$ is described hereinbelow in regard to the utility functions determined at every point for I^+ and I^-. We firstly consider the case of the 0 level curve. We ask the expert to provide the value of R_j for which $U(R_{i,j}^{K+}(R_j))=0$:

Q5: Onward of (if $\varepsilon_j=1$)/up to (if $\varepsilon_j=-1$) what value of R_j do you think that the variables R_j and $R_q=R_q^{kq}$ for $q\in A^+\backslash\{j\}$ compensate entirely for $R_i= R_i^k$ and $R_q=s_q$ for $q\in A^-\backslash\{i\}$?

We write $R_j^{iK,0}$ for this value. We must have:

(4-[C2-R1]) $$V_i^k + V_j(R_j^{iK,0}) + \Sigma_{q\in A+\backslash\{j\}} V_q^{kq} = 0$$

If the expert prefers to reason with regard to the −1 level curve, we ask him to provide the value of R_j for which $U(R_{i,j}^{K+}(R_j))= -1$:

Q5': Onward of (if $\varepsilon_j=1$)/up to (if $\varepsilon_j=-1$) what value of R_j do you think that the variables R_j and $R_q=R_q^{kq}$ for $q\in A^+\backslash\{j\}$ compensate entirely for $R_i= R_i^k$ and $R_q=s_q$ for $q\in A^-\backslash\{i\}$?

We write $R_j^{iK,0}$ for this value. We must have:

(4'-[C2-R1]) $$V_i^k + V_j(R_j^{iK,-}) + \Sigma_{q\in A+\backslash\{j\}} V_q^{kq} = -1$$

We add the variable j to the set D^+, and the set $I^-(j)\backslash D^-$ to D^-. If $D^+\neq I^+$, we return to step C2-R1-8.

C2-R1-9 - Determination of the parameters in the compensation situations: we shall now refer to the equations obtained. Relation (1-[C1-R1]) gives the value of s_q for all q. We set these equations aside, since this amounts to not considering $V_q(s_q)$ as an unknown. Thus, the unknowns are V_i^k for $i\in A^-$, $k\in\{-p'_i-1,\ldots,-1\}$ and V_j^k for $j\in A^+$, $k\in\{1,\ldots,p_j+1\}$. This gives $\Sigma_{q\in\{1,\ldots,n\}}$ (p_q+1) unknowns. On the other hand, we have the relation (3-[C2-R1]) for all $j\in A^+$, $k\in\{1,\ldots,p_j+1\}$, and the relation (4-[C2-R1]) or (4'-[C2-R1]) for all $j\in A^-$ and $k\in\{-p'_j-1,\ldots,-1\}$. In total

this gives $\Sigma_{q\in\{1,\ldots,n\}}$ (p_q+1) equations. To this must be added the conditions on the type of compensation. In the general case, this amounts to introducing a certain number of equations and inequalities in real variables and {0,1}, as described hereinbelow with reference to the determination of the particular point. We therefore have as many unknowns as equations. This therefore allows us to calculate a unique solution, if it exists. We can also add the following constraints:

$V_j^1 \le V_j^2 \le \ldots \le V_j^{pj+1}\ \forall\ j\in I^+$ and $V_i^{-1} \ge\ V_i^{-2} \ge \ldots \ge V_i^{-p'i-1}\ \forall\ i\in I^-$

We can solve a linear problem by minimizing the sum of errors over all the equations. If we insist that, out of all the inequalities in (2-[C1-R1]), one is an equality, we obtain a linear integer program.

<u>C2-R1-10 to C2-R1-12:</u> These steps are strictly identical to steps C1-R1-9 to C1-R1-11.

We will now examine the general case for which we assume that we can have a mixture between the cases $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_3$.

We firstly consider the utility functions determined at the extremities. The utility functions are determined as and when required. During the running of the algorithm, we denote by D^+ the set of variables R_j for which the positive part of the utility function V_j is determined, and by D^- the set of variables R_i for which the positive part of the utility function V_i is determined. At the start of the algorithm we have $D^+=\varnothing$ and $D^-=\varnothing$.

We assume that, for all i, the variable R_i satisfies the following hypothesis:

H-[C1-R*]: For R_{-i} fixed to s_{-i}, the compensation becomes completely impossible beyond a certain value of R_i.

Stated otherwise, a very bad value of R_i can no longer be compensated at all in respect of the neutral values (equal to the thresholds) along the other variables. This condition implies the following relation:

(1-[C1-R*]) $\quad \forall i, \quad V_{i,*} < -1$

We shall now give the details of the algorithm. The steps are numbered by a label always beginning with C1-R*1. The title C1 signifies that we are in case 1 (that is to say that we determine the utility functions solely at the extremities) while R* signifies that the variables may belong to any of the three compensation cases R1, R2 or R3.

The steps depend in part on the type of compensation considered (R1, R2 or R3). When a step depends on the type of compensation, we place a reminder of the corresponding compensation between brackets at the end of the step number. The process is based on the following steps:

C1-R*-1 - Definition of the reference thresholds $s_1, \ldots, s_n$: This step is strictly identical to step C1-R1-1.

C1-R*-2 - Definition of the values R_j^* for $j \in I^+$: This step is strictly identical to step C1-R1-2.

Determination of the V_i for $i \in \Re_1$: The steps whose number terminates with a square bracket [1] are specific to case R1 ($i \in \Re_1$). They will be split into cases R2 and R3 respectively with square brackets [2] and [3]. The steps that do not carry any square bracket are generic and are not repeated in cases R2 and R3.

C1-R*-3[1] - Definition of the values $R_{k,*}$ for $k \in I^-$: This step is strictly identical to step C1-R1-2.

C1-R*-4 - Determination of the utility function V_k between $R_{k,*}$ and s_k for $k \in I^-$: This step is strictly identical to step C1-R1-4.

C1-R*-5 - Determination of the utility function V_j between s_j and R_j^* for $j \in I^+$: This step is strictly identical to step C1-R1-5.

C1-R*-6[1] - Determination of reference $i \in \mathfrak{R}_1 \backslash D^-$: The questionnaire will be based on a particular variable from among $\mathfrak{R}_1 \backslash D^-$. This step is identical to step C1-R1-6.

C1-R1-7[1] - Determination of $V_{i,*}$ and V_j^* for all $j \in I^+(i) \backslash D^+$: Let $j \in I^+(i) \backslash D^+$. This step is identical to step C1-R1-7. By virtue of the question Q5[C1-R1], we determine $R_{i,j}^0$. As in step C1-R1-7, we have:

(2-[C1-R*]) $$V_{i,*} = \wedge_{(A+,A-) \in I / i \in A-} -(1+\Sigma_{j \in A+ \cap D+} V_j^*)/(1-\Sigma_{j \in A+} \lambda_i(R_{i,j}^0))$$

The expression for V_j^* for all $j \in I^+(i) \backslash D^+$ is:

(3-[C1-R*])

$$V_j^* = -\lambda_i(R_{i,j}^0) \times \wedge_{(A+,A-) \in I / i \in A-} -(1+\Sigma_{j \in A+ \cap D+} V_j^*)/(1-\Sigma_{j \in A+} \lambda_i(R_{i,j}^0))$$

We add the variable i to the set D^- and the set $I^+(i) \backslash D^+$ to D^+. If $D^- \neq I^-$ we return to a step C1-R1-6[1], C1-R1-6[2] or C1-R1-6[3].

C1-R*-8[1] - Determination of $V_{k,*}$ for $k \in \mathfrak{R}_1 \cap (I^- \backslash D^-)$ such that $I^+(k) \subset D^+$: For all $k \in \mathfrak{R}_1 \cap (I^- \backslash D^-)$ such that $I^+(i) \subset D^+$, we proceed exactly as in step C1-R1-8. The value of $V_{k,*}$ is then given by formula (6-[C1-R1]).

We then proceed to the determination of the V_i for $i \in \mathfrak{R}_2$ as follows.

C1-R*-3[2] - Definition of the values $R_{k,*}$ for $k \in I^-$: Below (if $\varepsilon_k=1$)/onward (if $\varepsilon_k=-1$) of $R_{k,*}$, the utility

function V_k remains stuck at the value $V_{k,*}$ and no longer decreases. In contradiction to compensation of the type R1, there is no limit behavior in the variable R_k beyond $R_{k,*}$ with a compensation of the type R2. To determine $R_{k,*}$, the expert answers the following question:

Q1[C1-R*]: Onward of (if $\varepsilon_k=1$)/below (if $\varepsilon_k=-1$) what value of R_k do you wish to no longer further penalize the compensation?

<u>**C1-R*-6[2] - Determination of reference $i \in \Re_2 \backslash D^-$:**</u> The questionnaire will be based on a particular variable out of $\Re_2 \backslash D^-$. This step is identical to step C1-R1-6. According to the condition given previously for R2, we have:

(4-[C1-R*]) $\exists (A^+,A^-) \in I$ **such that** $i \in A^-$**, we have** $0 < V_{i,*} + \Sigma_{j \in A+} V_j^*$

We ask the expert the pair (A^+,A^-) for which the previous inequality is satisfied.

Q2[C1-R*]: In which compensation situation, regardless of the value of R_i (even very bad), do we compensate entirely for sufficiently good values of the other variables?

<u>**C1-R*-7[2] - Determination of $V_{i,*}$ and V_j^* for all $j \in A^+ \backslash D^+$:**</u> For all $j \in A^+ \backslash D^+$, we ask the following question:

Q3[C1-R*]: For $R_{-\{i,j\}}$ fixed at $s_{-\{i,j\}}$, the variable R_j at the value R_j^* does it compensate at least a little for the variable R_i at the value $R_{i,*}$?

- If the reply to Q3[C1-R*] is positive, then this signifies that $V_{i,*}+V_j^*>-1$. So consider the vector of variables $R(R_j)$ such that: $(R(R_j))_j=R_j$, $(R(R_j))_i=R_{i,*}$ and $(R(R_j))_k=s_k$ for $k \notin \{i,j\}$. We have: $U(R(R_j))=V_j(R_j)+V_{i,*}$. For $R_j=s_j$, we have: $U(R(s_j))=V_{i,*}<-1$ according to (1-[C1-R*]), and for $R_j=R_j^*$, we have $U(R(R_j^*))=V_j^*+V_{i,*}>-1$ according to

the reply to the question Q5[C1-R2]. From this we deduce that there exists R_j^0 lying between s_j and R_j^* such that $U(R(R_j^*))=-1$. As $V_{i,*}<-1$, we have $R_j^0<s_j$ and therefore $\lambda_j(R_j^0)>0$. To determine R_j^0, the expert answers the following question:

Q3[C1-R*]: For $R_{-\{i,j\}}$ fixed at $s_{-\{i,j\}}$, up to (if $\varepsilon_j=1$)/onward of (if $\varepsilon_j=-1$) what value of R_j do you think that the variable R_i at the value $R_{i,*}$ is no longer compensated for at all?

As $U(R(R_j^0))= -1$, we have:

(5-[C1-R*]) $$V_{i,*} + \lambda_j(R_j^0)\ V_j^* = -1$$

- If the reply to Q3[C1-R*] is negative, then this signifies that $V_{i,*}+V_j^*\leq-1$. So consider the vector of variables $R(R_i)$ such that $(R(R_i))_i=R_i$, $(R(R_i))_j=R_i^*$ and $(R(R_i))_k=s_k$ for $k\notin\{i,j\}$. We have: $U(R(R_i))=V_i(R_i)+V_j^*$. For $R_i=s_i$, we have: $U(R(s_i))=V_j^*\geq0$, and for $R_i=R_{i,*}$, we have: $U(R(R_{i,*}))=V_{i,*}+V_j^*\leq-1$ according to the reply to question Q5[C1-R2]. From this we deduce that there exists R_i^0 lying between s_i and $R_{i,*}$ such that $U(R(R_i^0))=-1$. To determine R_i^0, the expert answers the following question:

Q3'[C1-R*]: For $R_{-\{i,j\}}$ fixed at $s_{-\{i,j\}}$, onward of (if $\varepsilon_i=1$)/below (if $\varepsilon_i=-1$) what value do you think that R_i can no longer be compensated at all by R_j at the value R_j^*?

As $U(R(R_i^0))= -1$, we have:

(5'-[C1-R*]) $$V_j^* + \lambda_i(R_i^0)\ V_{i,*} = -1$$

We put: $\tau_{ij}=\lambda_i(R_i^0)$ if $V_{i,*}+V_j^*\leq-1$, and $\tau_{ij}=1/\lambda_j(R_j^0)$ if $V_{i,*}+V_j^*>-1$. We moreover put $T_{ij}=1$ if $V_{i,*}+V_j^*\leq-1$, and $T_{ij}=1/\lambda_j(R_j^0)$ if $V_{i,*}+V_j^*>-1$. According to (4-[C1-R2]) and (4'-[C1-R2]), for all $j\in A^+\backslash D^+$, we have:

(5''-[C1-R*]) $$V_j^* + \tau_{ij}\ V_{i,*} = -T_{ij}$$

According to (4-[C1-R*])

$$V_{i,*}\ (1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij}\ ) - \Sigma_{j\in A+\backslash D+}\ T_{ij} + \Sigma_{j\in A+\cap D+}\ V_j^* > 0$$

We consider the equality when the right-hand side equals a fixed positive number κ. Hence:

(6-[C1-R*]) $V_{i,*} = (\kappa+\Sigma_{j\in A+\backslash D+} T_{ij} - \Sigma_{j\in A+\cap D+} V_j^*) / (1-\Sigma_{j\in A+\backslash D+} \tau_{ij})$

With (5''-[C1-R*]), we obtain, for all $j\in A^+\backslash D^+$:

(7-[C1-R*]) $V_j^* = - T_{ij} - \tau_{ij} \times (\kappa+\Sigma_{k\in A+\backslash D+} T_{ik} - \Sigma_{k\in A+\cap D+} V_k^*) / (1-\Sigma_{k\in A+\backslash D+} \tau_{ik})$

We add the variable i to the set D^- and the set $A^+\backslash D^+$ to D^+. If $D^-\neq I^-$, we return to a set C1-R1-6[1], C1-R1-6[2] or C1-R1-6[3].

<u>**C1-R*-8[2] - Determination of $V_{k,*}$ for $k\in\Re_2 \cap I^-\backslash D^-$ such that $I^+(k)\subset D^+$:**</u> For all $k\in\Re_2 \cap I^-\backslash D^-$ such that $I^+(i)\subset D^+$, we can determine the value of $V_{k,*}$. We ask the question Q2[C1-R*] to determine the pair (A^+,A^-) for which the relation (4-[C1-R*]) is satisfied. According to (4-[C1-R*]), we have:

$V_{k,*} + \Sigma_{j\in A+} V_j^* > 0$

We obtain:

$\mathbf{V_{k,*} = \kappa - \Sigma_{j\in A+} V_j^*}$

We then proceed to the determination of the V_i for $i\in\Re_3$:

<u>**C1-R*-3[3] - Definition of the values $R_{k,*}$ for $k\in I^-$:**</u> This step is strictly identical to step C1-R*-3[2].

<u>**C1-R*-6[3] - Determination of reference $i\in\Re_3\backslash D^-$:**</u> The questionnaire will be based on a particular variable out of $\Re_3\backslash D^-$. This step is identical to step C1-R1-6.

<u>**C1-R*-7[3] - Determination of $V_{i,*}$ and V_j^* for all $j\in I^+(i)\backslash D^+$:**</u> For all $j\in I^+(i)\backslash D^+$, we proceed exactly as for the start of C1-R*-7[2]. For all $j\in I^+(i)\backslash D^+$, we culminate in the relation (5''-[C1-R*]).

The second relation corresponding to the case R3 gives for all $(A^+,A^-)\in I$ with $i\in A^-$:

$$V_{i,*}\ (1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij})\ -\ \Sigma_{j\in A+\backslash D+}\ T_{ij}\ +\Sigma_{j\in A+\cap D+}\ V_j^*\ \leq\ 0$$

Hence:

$$V_{i,*}\ \leq\ (\Sigma_{j\in A+\backslash D+}\ T_{ij}\ -\Sigma_{j\in A+\cap D+}\ V_j^*)\,/\,(1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij})$$

and:

$$V_{i,*}\ \leq\ \wedge_{(A+,A-)\in I/i\in A-}\ \ (\Sigma_{j\in A+\backslash D+}\ T_{ij}\ -\Sigma_{j\in A+\cap D+}\ V_j^*)\,/\,(1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij})$$

We take the equality:

(8-[C1-R*])

$$V_{i,*}\ =\ \wedge_{(A+,A-)\in I/i\in A-}\ (\Sigma_{j\in A+\backslash D+}\ T_{ij}\ -\Sigma_{j\in A+\cap D+}\ V_j^*)\,/\,(1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij})$$

For the pair $(A^+,A^-)\in I$ that achieves the minimum, we have: $V_{i,*}\ +\Sigma_{j\in A+}\ V_j^*\ =\ 0$. This implies that for this pair, we have in particular: $V_{i,*}\ +\Sigma_{j\in A+}\ V_j^*\ >\ -1$. From this we deduce that the first relation corresponding to the case R3 is satisfied. Consequently, $V_{i,*}$ given by (8-[C1-R*]) satisfies the conditions of the case R3.

With (5''-[C1-R*]), we obtain, for all $j\in I^+(i)\backslash D^+$:

(9-[C1-R*]) $V_j^*\ =\ -\ T_{ij}\ -\tau_{ij}\ \times\ \wedge_{(A+,A-)\in I\ /\ i\in A-}\ \ (\Sigma_{j\in A+\backslash D+}\ T_{ij}\ -\Sigma_{j\in A+\cap D+}\ V_j^*)\,/\,(1-\Sigma_{j\in A+\backslash D+}\ \tau_{ij}\ )$

We add the variable i to the set D^- and the set of $I^+(i)\backslash D^+$ to D^+. If $D^-\neq I^-$, we return to a step C1-R1-6[1], C1-R1-6[2] or C1-R1-6[3].

<u>C1-R*-8[3] - Determination of $V_{k,*}$ for $k\in\Re_3\cap I^-\backslash D^-$ such that $I^+(k)\subset D^+$:</u> For all $k\in\Re_3\cap I^-\backslash D^-$ such that $I^+(i)\subset D^+$, we can determine the value of $V_{k,*}$. The second relation corresponding to the case R3 gives:

$$V_{k,*}\ \leq\ \wedge_{(A+,A-)\in I/k\in A-}\ \ -\Sigma_{j\in A+}\ V_j^*$$

As previously, we take the equality:

$$V_{k,*}\ =\ \wedge_{(A+,A-)\in I/k\in A-}\ \ -\Sigma_{j\in A+}\ V_j^*$$

For the pair $(A^+,A^-)\in I$ achieving the minimum, we have: $V_{k,*}\ +\Sigma_{j\in A+}\ V_j^*\ =\ 0$. From this we deduce that the first relation corresponding to the case R3 is satisfied. Consequently, $V_{k,*}$ given by the previous formula satisfies the conditions of case R3.

We then proceed to the determination of the V_j for $j \in I^+ \setminus D^+$, as follows.

C1-R*-9 - Determination of V_j^* for all $j \in I^+ \setminus D^+$: On completion of steps C1-R*-7, if $I^+ \neq D^+$, this is necessarily due to a case R2, that is to say to a pair $(A^+, A^-) \in I$ with $j \in A^+$, for which all the $i \in A^-$ corresponding to the case R2. Let therefore $j \in I^+ \setminus D^+$. Let $(A^+, A^-) \in I$ with $j \in A^+$. We ask the expert, echoing the step C1-R1-2, for an index $i \in A^-$ on which a question will be asked. We then proceed exactly as at the start of step C1-R*-7[2]. We culminate in relation (5''-[C1-R*]). This gives the expression for V_j^* :

$$\textbf{(9-[C1-R*])} \qquad V_j^* = - T_{ij} - \tau_{ij} \times V_{i,*}$$

We will now examine how the determination of the parameters goes outside of the framework of compensation.

C1-R*-10 -- Determination of V_j^* for $j \notin I^+$: This step is strictly identical to step C1-R1-9.

C1-R*-11 -- Determination of V_i beyond $R_{i,*}$ for $i \in \mathfrak{R}_1$: In the framework of a compensation of type R1, it is necessary to ensure that no more compensation is possible below (if $\varepsilon_i=1$)/beyond (if $\varepsilon_i=-1$) a certain value of R_i. It is for this reason that we saw that it is necessary to introduce a point $R_{i,**}$ below (if $\varepsilon_i=1$)/beyond (if $\varepsilon_i=-1$) $R_{i,*}$. This step is strictly identical to step C1-R1-10.

C1-R*-12 -- Determination of $R_{k,*}$ and $V_{k,*}$ for $k \notin I^-$: For $k \notin I^-$, the variable R_k is never presumed to be compensated for by other variables. We therefore assume that the variables that are not a priori among the variables presumed to be compensated according to the expert, belong to the case R1, that is to say to the

most restrictive case. We then proceed exactly as in step C1-R1-11.

To determine the utility functions at every point for I^+ and I^-, the procedure is similar to what was described above with reference to figure 15. We shall not describe it in detail. It involves generalizing steps C2-R1-6 and C2-R1-8. We then seek to determine V_i^k for all $i \in I^-$, $k \in \{-p'_i -1, \ldots, -1\}$ and for all $i \in I^+$, $k \in \{1, \ldots, p_i+1\}$. These are the unknowns in question.

Let $i, j \in N$ and $A \subseteq N \setminus \{i,j\}$. Let R be the vector whose coordinates are the following: $R_i = R_i^k$, R_j a nonfixed value, $R_q = R_q^{kq}$ for all $q \in A \setminus j$, and $R_q = s_q$ for all $q \in N \setminus (A \cup \{i,j\})$. We have $U(R) = V_i^k + V_j(R_j) + \Sigma_{q \in A \setminus \{j\}} V_q^{kq}$. The vector R therefore employs the utility V_i^k. In order to have a relation satisfied by V_i^k, we therefore seek to have R_j such that $U(R)=0$ or $U(R)= -1$. We therefore seek indices $\{k_q\}_{q \in A \setminus j}$ such that when R_j goes from $R_{j,*}$ to s_j, we are certain that U(R) crosses one of the two 0 or -1 level curves. We describe here the manner in which the indices $\{k_q\}_{q \in A \setminus j}$ are determined.

For each $i \in I^-$, $k \in \{-p'_i -1, \ldots, -1\}$ and each $i \in I^+$, $k \in \{1, \ldots, p_i+1\}$, we use this technique to determine a relation satisfied by V_i^k. We begin with those for which we do not need to search for the indices $\{k_q\}_{q \in A \setminus j}$ numerically (that is to say those for which we know a priori some $\{k_q\}_{q \in A \setminus j}$ for which U(R) crosses one of the two 0 or -1 level curves).

To perform the determination of the particular point, the difficulty is that we do not yet know the values of the utilities V_q^{kq}. Under these conditions, how does one determine the best indices $\{k_q\}_{q \in A \setminus j}$ such that U(R) definitely crosses one of the two level curves? The idea is, based on the elements available to us, to determine the best indices $\{k_q\}_{q \in A \setminus j}$ such that it is entirely possible for U(R) to cross one of the two

level curves, that is to say that there is no contra-indication from the elements available to us. The elements available to us are grouped into a set denoted Ψ. These are the relations available to us regarding the unknowns. We denote by Ψ the set of equalities originating from the particular points already determined, and from the subsequent relations originating from the type of compensation chosen:

- **Case where $i \in \mathfrak{R}_1$**. For all $(A^+,A^-) \in I$ such that $i \in A^-$, we have $V_{i,*} + \Sigma_{j \in A+} V_j^* \leq -1$. Moreover, in order for $R_{i,*}$ to correspond to its definition given previously, at least one of these inequalities must be satisfied with an equality. This is modeled with the aid of a linear integer problem. This gives:

For all $(A^+,A^-) \in I$ such that $i \in A^-$:

$$V_{i,*} + \Sigma_{j \in A+} V_j^* = -1 - E_{i,A+}$$
$$E_{i,A+} \geq 0$$
$$\varepsilon_{i,A+} \in \{0,1\}$$
$$\varepsilon_{i,A+} \leq E_{i,A+}/\delta$$
$$\varepsilon_{i,A+} \geq (E_{i,A+} - \delta)/E_{max}$$
$$\Sigma_{(A+,A-) \in I \text{ such that } i \in A-} (1 - \varepsilon_{i,A+}) \geq 1$$

where δ is a very small number (for example $\delta = 10^{-8}$) and E_{max} is an a priori upper bound of the $E_{i,A+}$ (for example $E_{max} = 10^8$). The variables $\varepsilon_{i,A+}$ are integers. The relation $\varepsilon_{i,A+} \leq E_{i,A+}/\delta$ implies that $\varepsilon_{i,A+} = 0$ once $E_{i,A+} < \delta$. The relation $\varepsilon_{i,A+} \geq (E_{i,A+} - \delta)/E_{max}$ implies that $\varepsilon_{i,A+} = 1$ once $E_{i,A+} > \delta$. In all, $\varepsilon_{i,A+} = 0$ if $V_{i,*} + \Sigma_{j \in A+} V_j^* = -1$ and $\varepsilon_{i,A+} = 1$ if $V_{i,*} + \Sigma_{j \in A+} V_j^* < -1$. From this we deduce that $\Sigma_{(A+,A-) \in I \text{ such that } i \in A-} (1 - \varepsilon_{i,A+})$ gives the number of $(A^+,A^-) \in I$ for which we have equality in $V_{i,*} + \Sigma_{j \in A+} V_j^* \leq -1$. The relation $\Sigma_{(A+,A-) \in I \text{ such that } i \in A-} (1 - \varepsilon_{i,A+}) \geq 1$ indicates therefore that at least one of these inequalities is satisfied with an equality.

- **Case where $i \in \mathfrak{R}_2$**. For all $(A^+,A^-) \in I$ such that $i \in A^-$ we have $V_{i,*} + \Sigma_{j \in A+} V_j^* > 0$. We write this in the following manner: for all $(A^+,A^-) \in I$ such that $i \in A^-$ we have $V_{i,*} + \Sigma_{j \in A+} V_j^* \geq \gamma$ (with γ very small). Moreover, in order for $R_{i,*}$ to correspond to its definition given previously, at least one of these

inequalities must be satisfied with an equality. We model this with the aid of a linear integer problem. This way of doing things is similar to that of the previous case.

For all $(A^+,A^-)\in I$ such that $i\in A^-$:

$$V_{i,*} + \Sigma_{j\in A+} V_j^* = \gamma + E_{i,A+}$$
$$E_{i,A+} \geq 0$$
$$\varepsilon_{i,A+} \in \{0,1\}$$
$$\varepsilon_{i,A+} \leq E_{i,A+}/\delta$$
$$\varepsilon_{i,A+} \geq (E_{i,A+} - \delta)/E_{max}$$
$$\Sigma_{(A+,A-)\in I \text{ such that } i\in A-} (1-\varepsilon_{i,A+}) \geq 1$$

❑ **Case where $i\in\Re_3$.** No additional condition is imposed other than the conditions on $\Re_3$. This gives:

For all $(A^+,A^-)\in I$ such that $i\in A^-$:

$$-1< V_{i,*} + \Sigma_{j\in A+} V_j^* \leq 0$$

We saw previously that:

$\exists R_j\in [s_j,R_j^*]$ such that $U(R) = 0 \Leftrightarrow V_i^k \in [-V_j(R_j) -\Sigma_{q\in A} V_q^{kq} , -\Sigma_{q\in A} V_q^{kq}]$ (1)

$\exists R_j\in [s_j,R_j^*]$ such that $U(R) = -1 \Leftrightarrow V_i^k \in [-1-V_j(R_j) -\Sigma_{q\in A} V_q^{kq} , -1-\Sigma_{q\in A} V_q^{kq}]$ (2)

We therefore want (1) or (2) to be satisfied: Let:

$L' = \{ (v_j^*,v_i^k,\{v_q^{kq}\}_{q\in A})$ such that $\Psi\cup\{V_j^*=v_j^*\}\cup\{V_i^k=v_i^k\}\cup\{V_q^{kq}=v_q^{kq}\}_{q\in A} \cup \{(1) \text{ or } (2)\} \neq\varnothing\}$

$L= \{ (v_j^*,\{v_q^{kq}\}_{q\in A})$ such that $\exists\, v_i^k$ with $(v_j^*,v_i^k,\{v_q^{kq}\}_{q\in A})\in L \}$

The set L provides the values of the variables $v_j^*,v_i^k,\{v_q^{kq}\}_{q\in A}$ that are compatible with Ψ, and (1) or (2). Stated otherwise, if $L\neq\varnothing$ then (1) or (2) will be achievable and U(R) will cut one of the two 0 or −1 level curves for a value of R_j. We want $L\neq\varnothing$. To go further, we want to maximize the span of the possible v_i^k. Specifically, the more possible values of V_i^k there are, the more margin we will have as regards (1) or (2). Let:

$r_L=\wedge_{(vj*,\{vqkq\}q\in A)\in L} (\vee_{(vj*,vik,\{vqkq\}q\in A)\in L'} V_i^k -\wedge_{(vj*,vik,\{vqkq\}q\in A)\in L'} v_i^k)$

We therefore want to choose $\{k_q\}_{q\in A}$ so as to maximize r_L. This number is not easy to calculate. We seek an approximation thereof that is simple to calculate.

Let

$\underline{V}_j^{\bullet} = \wedge_{vj^* \text{ such that } \Psi\cup\{Vj^*=vj^*\}\neq\emptyset}\ v_j^{*}$ and $\overline{V}_j^{\bullet} = \vee_{vj^* \text{ such that } \Psi\cup\{Vj^*=vj^*\}\neq\emptyset}\ v_j^{*}$

$\underline{V}_i^{k} = \wedge_{vik \text{ such that } \Psi\cup\{Vik=vik\}\neq\emptyset}\ v_i^{k}$ and $\overline{V}_i^{k} = \vee_{vik \text{ such that } \Psi\cup\{Vik=vik\}\neq\emptyset}\ v_i^{k}$

$\underline{\lambda} = \wedge_{\{vqkq\}q\in A \text{ such that } \Psi\cup\{Vqkq=vqkq\}q\in A\neq\emptyset}\ \Sigma_{q\in A}\ V_q^{kq}$

and $\overline{\lambda} = \vee_{\{vqkq\}q\in A \text{ such that } \Psi\cup\{Vqkq=vqkq\}q\in A\neq\emptyset}\ \Sigma_{q\in A}\ V_q^{kq}$

Let:

$M' = \{(v_j^*, v_i^k, \{v_q^{kq}\}_{q\in A})$ such that $v_j^* \in [\underline{V}_j^{\bullet}, \overline{V}_j^{\bullet}]$, $v_i^k \in [\underline{V}_i^{k}, \overline{V}_i^{k}]$, $\Sigma_{q\in A}\ V_q^{kq} \in [\underline{\lambda}, \overline{\lambda}]$ and (1) or (2)$\}$

$M = \{(v_j^*, \{v_q^{kq}\}_{q\in A})$ such that $\exists\ v_i^k$ with $(v_j^*, v_i^k, \{v_q^{kq}\}_{q\in A}) \in M\ \}$

We have $L' \subseteq M'$ and $L \subseteq M$. M and M' are therefore over approximations of the sets L and L'. Let:

$r_M = \wedge_{(vj^*,\{vqkq\}q\in A)\in M}\ (\ \vee_{(vj^*,vik,\{vqkq\}q\in A)\in M'}\ v_i^k - \wedge_{(vj^*,vik,\{vqkq\}q\in A)\in M'}\ v_i^k\ )$

As $L' \subseteq M'$ and $L \subseteq M$, it follows that:

$r_M \geq r_L$

The number r_M is very easy to calculate. Indeed, we put:

$\rho=0$ if we consider relation (1)

$\rho=1$ if we consider relation (2)

Firstly, we observe that:

$r_M = \min\ \{\ r(v_j^*, \lambda),\ v_j^* \in [\underline{V}_j^{\bullet}, \overline{V}_j^{\bullet}],\ \ \lambda \in [\underline{\lambda}, \overline{\lambda}]\ \}$

where:

$r(v_j^*, \lambda) = |\ [\underline{V}_i^{k}, \overline{V}_i^{k}] \cap [-v_j^* - \lambda - \rho, -\lambda - \rho]\ |$

For an interval $T=[T_0, T_1]$, the length $|T|$ of this interval is defined by $T_1 - T_0$. We have:

$r_M = 0 \iff \overline{V}_i^{k} \leq -\underline{V}_j^{\bullet} - \underline{\lambda}$ or $\underline{V}_i^{k} \geq -\overline{\lambda}$

Moreover, it is easy to see that:

$r_M = r(\underline{V}_j^{\bullet}, \underline{\lambda}) \wedge r(\underline{V}_j^{\bullet}, \overline{\lambda})$

The indices $\{k_q\}_{q\in A+\backslash j}$ considered are those which maximize r_L (if we wish to solve the exact problem) or r_M (if we are content with the approximate problem).